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Brad Fulton

Educator of the Year

- ♦ Consultant
- ♦ Educator
- ♦ Author
- ♦ Keynote presenter
- ♦ Teacher trainer
- ♦ Conference speaker

Known throughout the country for motivating and engaging teachers and students, Brad has co-authored over a dozen books that provide easy-to-teach yet mathematically rich activities for busy teachers while teaching full time for over 30 years. In addition, he has co-authored over 40 teacher training manuals full of activities and ideas that help teachers who believe mathematics must be both meaningful and powerful.

Seminar leader and trainer of mathematics teachers

- ♦ 2005 California League of Middle Schools Educator of the Year
- ♦ California Math Council and NCTM national featured presenter
- ♦ Lead trainer for summer teacher training institutes
- ♦ Trainer/consultant for district, county, regional, and national workshops

Author and co-author of mathematics curriculum

- ♦ Simply Great Math Activities series: six books covering all major strands
- ♦ Angle On Geometry Program: over 400 pages of research-based geometry instruction
- ♦ Math Discoveries series: bringing math alive for students in middle schools
- ♦ Teacher training seminar materials handbooks for elementary, middle, and secondary school

Available for workshops, keynote addresses, and conferences

All workshops provide participants with complete, ready-to-use activities that require minimal preparation and give clear and specific directions. Participants also receive journal prompts, homework suggestions, and ideas for extensions and assessment.

Brad's math activities are the best I've seen in 38 years of teaching!
Wayne Dequer, 7th grade math teacher, Arcadia, CA

"I can't begin to tell you how much you have inspired me!"
Sue Bonesteel, Math Dept. Chair, Phoenix, AZ

"Your entire audience was fully involved in math!! When they chatted, they chatted math. Real thinking!"
Brenda McGaffigan, principal, Santa Ana, CA

"Absolutely engaging. I can teach algebra to second graders!"
Lisa Fellers, teacher

VanHiele Research on geometry

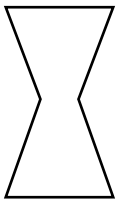
The Dutch mathematicians Dina and Pierre VanHiele developed the seminal model on the acquisition of geometric understanding in the 1950's. Though their findings have been validated and supported for decades, it has been slow to find its way into the American education system. In elementary and even in middle school, geometry is often overly simplified when students are asked to memorize content without exploring and developing it. Other times it is passed over entirely, or merely relegated to the final chapter of the book – a no-man's land where teachers rarely find time to venture.

Thus, for many students their first venture into the domain of geometry comes when they have to pass a high-level course in secondary school. This coupled with the fact that the part of the brain that is dedicated to geometry is not the same region that deals with numerical mathematics means that many students fail this required course.

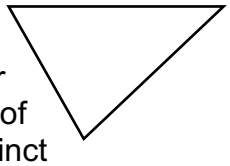
However, the solution to this problem is clear and straightforward. Students who are taught consistently through the VanHiele model are much more likely to develop the necessary skills to succeed in geometry.

Level 0: Visualization

Children **recognize shapes** by appearance: square, circle, rectangle. A child may call a sphere or cylinder a circle at this point, not distinguishing between 2D and 3D shapes. For example, a coin is a circle to children at this level. Students may apply the term hexagon to an octagon.

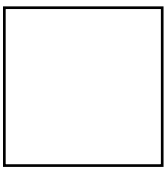


Similarly, if a shape does not fit their classification scheme, they may reject it. A rotated square is called a diamond or rhombus. An hourglass or bowtie shape may not be called a hexagon because it is not regular. A student may not be able to identify the base of a trinagle that has a horizontal side at the top and a vertex pointing downward.

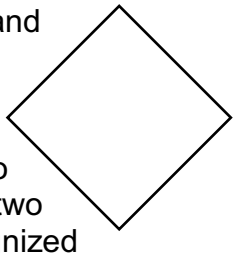


These students see shapes as separate classifications and ignore their interrelationships. For example, they don't see a rectangle as a subset of parallelograms. They often see a square and a rectangle as two distinct shapes instead of thinking of a square as a subset of the family of rectangles.

These misconceptions are often reinforced in primary grades. Textbooks and worksheets often provide limited and very basic examples for geometric terms. An illustration may have the word square beneath a single picture in which the sides of the square are parallel to the sides of the paper. Other examples may not be shown, so the student does not see a rotated square as a square.



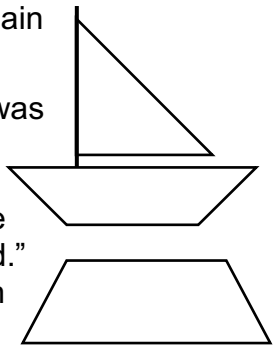
A shocking example occurred when I was working with some fourth and fifth grade classes. I put a square tile on a projector, but it landed as shown. I asked the students what the shape was, and many shouted that it was a diamond. A few frantically waved their hands, and when I called on them, they said it was a rhombus. When I rotated it parallel to the edges of the screen, they said it was a square. I repeated this in two fourth and two fifth grade classes. Only one student in over 100 recognized that it was a square regardless of orientation!



These students were taught to recognize squares as part of their kindergarten math standards, but even five years later, only 1% knew what a square was. I was so surprised that I wondered if my eighth graders could recognize a square when it was rotated. When I repeated the experiment with them, most called the rotated square a square. Somewhere in the interim between fourth and eighth grade, they had begun to move out of level 0 on the VanHiele scale.

This is called the “black swan paradox”. In Europe, all swans were white. Whether you were in the British Isles, western Europe, eastern Europe, or even Africa or Asia, swans were white. When swans were seen in Australia, they were black. This forced us to change our definition of what a swan is.

I met a teacher who divides her class into groups based on geometric shapes that hang over their tables. There is the triangle group and the circle group. But there is also the square group and the rectangle group. When students in the rectangle group are told to go get their materials, students in the square group remain seated.



I encountered a similar experience with my granddaughter when she was in the 5th grade. We were playing a game where we describe a picture while the other person tried to draw it. I wanted her to draw the sailboat shown here. I described the hull as a trapezoid with the longer base on the top. She said, “Oh, an upside-down trapezoid.” Clearly, she had only encountered trapezoids like the one shown in black.

My grandson while in kindergarten called any shape with more than four sides and octagon. I suspect that he had seen a picture of an octagon in class and noticed that it had more sides than a rectangle. That may have been all he noticed about it.

Many older students and even adults are at this level of geometric understanding. To move them beyond this stage, one good activity is the “This is/This isn’t” activity. Given a set of shapes, you could say, “This *is* a polygon,” or “This *is not* a polygon,” until students note the similarities and differences among them.

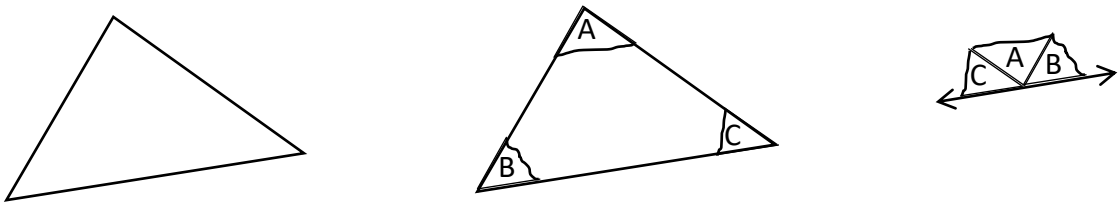
Level 1: Analysis

At this level, students will **focus on the properties** inherent in shapes. These students realize that a rotated square is still a square. The characteristics and properties of a shape take precedence over its appearance.

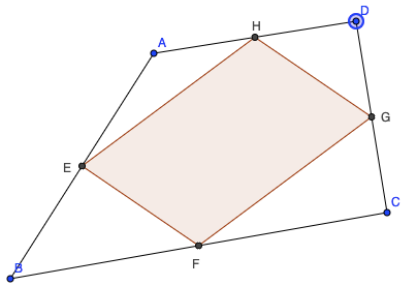
They will begin to define a square by its properties, though they may not be able to do this perfectly. They might say a square has four congruent sides and neglect the fact that it also has four congruent angles.

To develop this stage, educators should expose students to activities that will illustrate the properties of shapes.

- Create any triangle and cut it out. Remove the vertices and set them upon a common point. How many degrees are there? (180º)

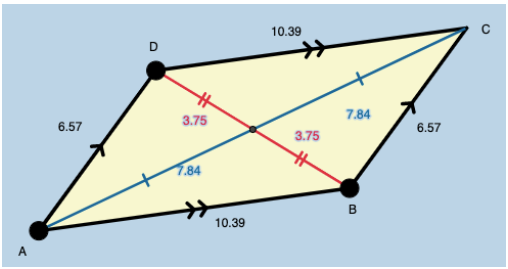


- Create any quadrilateral. Locate the midpoints of the side. Connect them to form a new quadrilateral. What is the name of this shape? (Parallelogram)



(Image from GeoGebra)

- Compare the diagonals of different quadrilaterals. What characteristics do they share?



(Image from GeoGebra)

Manipulative and computer-based activities are crucial in helping students focus on the properties of geometric shapes.

- www.geogebra.org/

Level 2: Abstraction

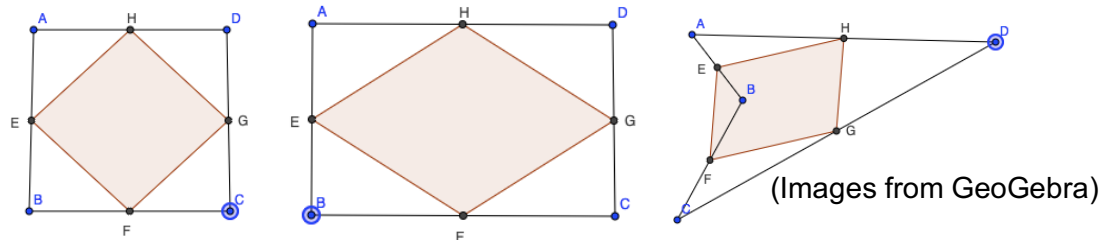
Students will begin to see how shapes relate to one another and can see that a square is therefore both a rhombus and a rectangle. They do this by seeing that properties of one shape may apply to another also.

They will begin to reason about shapes and their properties, though this is often based on **inductive reasoning** (recognition and generalizations of patterns and similarities based on observations). To develop this level of ability, lead the students to make a discovery such as the fact that the vertices of a quadrilateral add up to 360º. Then have them **test this repeatedly** with various types of regular and irregular quadrilaterals.

The deductive reasoning used at this stage is much like the thinking of a scientist. Observations are made, additional examples are tested, and a hypothesis (a conjecture in mathematics) is made. This hypothesis is then tested further to see if there are counterexamples.

For example, students can be asked to construct a quadrilateral with a straightedge. The teacher then asks students to compare their quadrilaterals with other students in their group. Some will make squares, some will make rectangles, and others may make rhombi, parallelograms or irregular quadrilaterals. The teacher encourages the students to notice that every student in the class constructed a unique quadrilateral.

The students are then asked to connect the midpoints of the sides. The student who constructed a square will get a square. The student with a rectangle now sees a rhombus inside it. And even the student with an irregular concave quadrilateral has a parallelogram.



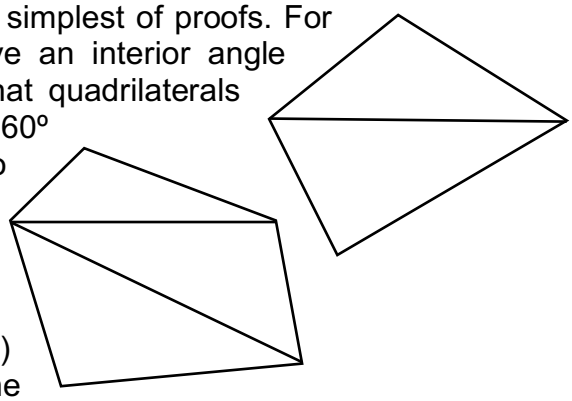
(Images from GeoGebra)

Students are asked to share their observations with their peers. Even the square and the rhombus are parallelograms by definition. They conclude that the midpoints of a quadrilateral form the vertices of a parallelogram. This has occurred with every student in the class – maybe 30 out of 30. They use their inductive reasoning (reasoning by observation and example) to conclude that the midpoints of *every* quadrilateral define the vertices of a parallelogram. However, they have not *proven* this to be true for the infinite number of quadrilaterals in the universe.

Although students at this stage of development show a high level of understanding, they fail to reason deductively or to understand the need for postulates, conjectures, and theorems. They follow hunches and intuition more than proof. Again, geometry software can be of great help in developing these generalizations because it allows students to quickly create and test multiple iterations of their shapes.

Level 3: Deduction

This is the classic stage of high school geometry. Students can reason deductively (based on accepted postulates and derived theorems). They can follow or create a deductive proof given certain initial conditions.



To help students in this stage, begin with the simplest of proofs. For example, if we accept that all triangles have an interior angle measurement of 180º, then we can prove that quadrilaterals must have an interior angle measurement of 360º since any quadrilateral can be divided into two triangles. Similarly, any pentagon can be subdivided into three triangles for an interior angle sum of 540º. Continuing this way, it can be shown algebraically that the formula for the interior angle sum of any polygon is 180(n–2) where n represents the number of sides. The quadrilateral has four sides and two interior triangles. The pentagon has five sides and three interior triangles. Thus, the number of interior triangles seems to be two less than the number of sides.

I find it helpful for students when I compare this stage to a court trial. We cannot base guilt and innocence on hunches and simple observations of patterns: “The last three people who got caught speeding had red cars, so if you have a red car, you are guilty of speeding.” In a court proceeding, guilt must be proven, even if it is obvious. We depend upon evidence such as fingerprints or DNA that cannot be refuted. Though in most cases, inductive reasoning will get us through, there are times when we want to be absolutely sure. Geometry and justice are like that.

Level 4: Rigor

At this level, we can explore beyond plane geometry. For example, lines of latitude are perpendicular to the equator but don’t produce parallel lines. Instead they converge in both directions due to the curvature of the earth’s surface.

We would also find more rigorous proofs at this level, such as proof by negation.

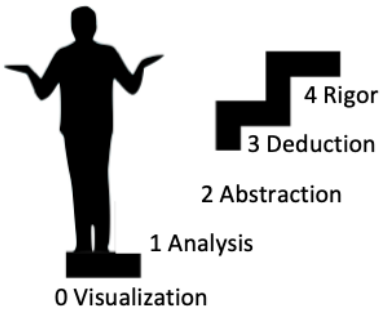
Sadly, though many students are at level 0, high school geometry is taught at levels 3 and 4. And unlike some subjects, students must proceed through these levels sequentially; they cannot skip steps and find success. It is best to imagine the five levels as rungs on a ladder or steps in a staircase. If steps are missing, it is nearly impossible for most students to climb to the levels we see in secondary geometry.

Fortunately, to some degree the movement from one rung to the next is not dependent solely on age but is accelerated by experiences. That means that as we provide these opportunities to students in elementary and middle school, they are more likely to find success in high school geometry.

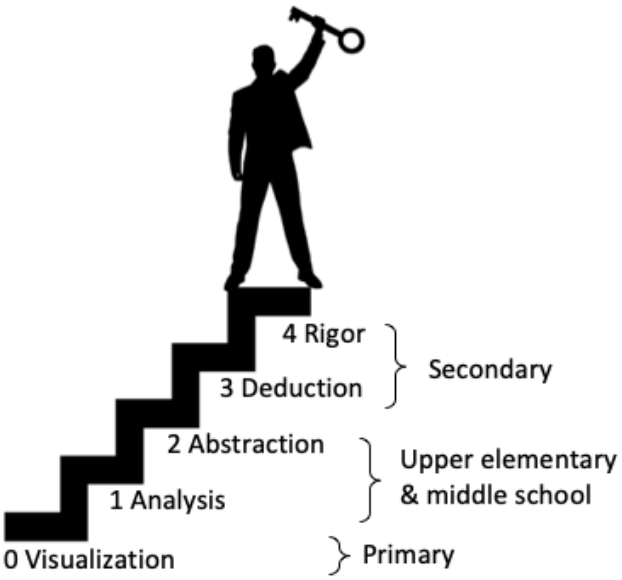
Sadly, American textbooks and programs are lacking in instructional activities at levels 1 (Analysis) and 2 (Abstraction). For example, in my state, the 8th grade geometry standards focus largely on volumes and surface area of cylinders, cones, pyramids, and spheres. Students are asked to calculate these given the proper equations. This is likely an arithmetic activity – or at best, an algebra task – that does nothing to enhance the student's understanding of geometry.

The following three diagrams should help us visualize the task before us. The first figure shows a staircase modeling what the VanHiele research has found.

This diagram, on the other hand, shows what we typically encounter in our state standards and textbook instruction.



Lastly, this diagram shows where these steps can be built in order to help high school students achieve the greatest success.

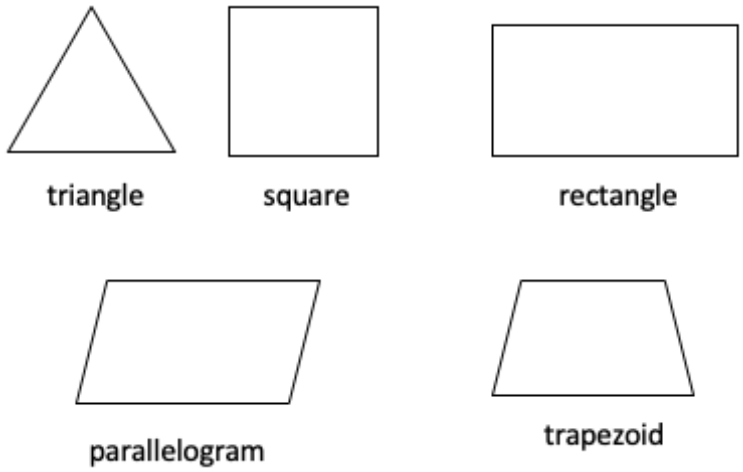


However, this is an ideal model. If we are tasked with teaching a secondary geometry class in which students are lacking the necessary upper elementary and middle school experiences, we must work with the cards we are dealt. In that case, we might consider a modification of our geometry course for students who need extra support. One option would be to offer a first semester that focuses on activities that develop analysis and abstraction followed by an abbreviated second semester of more formal geometry instruction. Of course, some content must be passed over in this case. The teacher would need to ensure that students are getting the geometry content that they will need for success in trigonometry and precalculus.

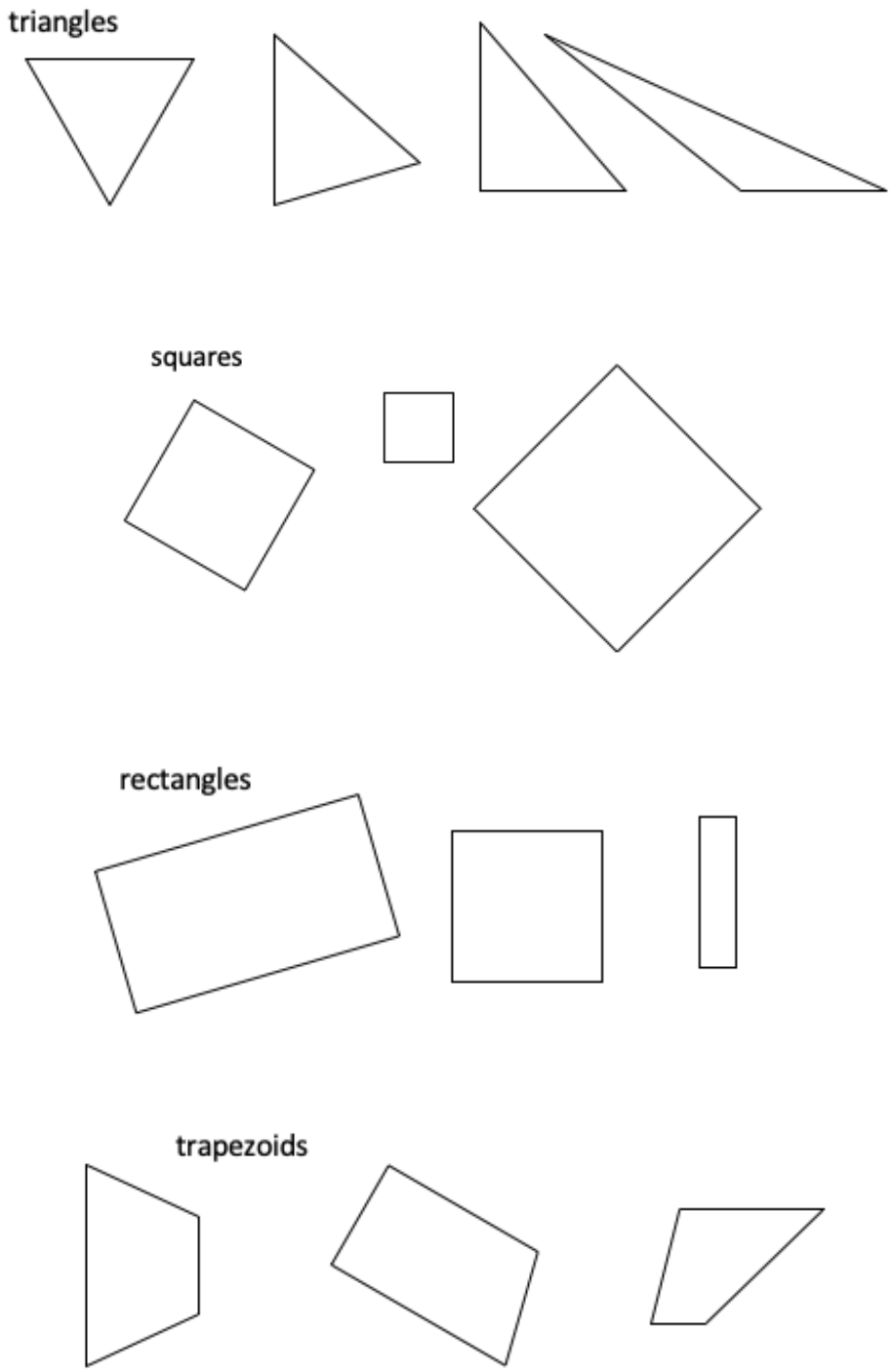
The purpose of this manual is to provide the teacher with examples of activities that will transition students out of level 0 and into level 1, from level 1 to level 2, and from level 2 toward level 3.

Setting the Foundation in Level 0: Visualization

We begin by looking at the transition from level 0: Visualization to level 1: Analysis. One strategy is to offer *multiple* examples in level 0 when we introduce a new concept. Instead of this model:



We use this approach. The more examples we use, the more likely students are to formulate correct assumptions about these shapes. With only one or a very limited number of examples, students form a prejudice of sorts; anything that doesn't fit their preconceived mold is rejected. When we offer multiple and varied samples, they form more realistic definitions and are now studying them with an eye toward **analysis** of the properties and characteristics they share.



Level 1: Analysis

One way to help students begin to move from sight recognition of shapes to an analysis of their characteristics is by using attribute shapes. These are kits that contain a variety of polygons. For example, in the quadrilateral family, there are square tiles, rectangular tiles, parallelograms, rhombi, trapezoids, kites, and irregular quadrilaterals.

The teacher can foster understanding of the properties by asking questions such as these:

“I am thinking of a quadrilateral with four congruent sides.”

Many students will grab the square, but some may notice that the rhombus also fits this condition. Then the teacher says, “I am thinking of a shape with four 90º angles.”

Again, the square will suffice, but so will the rectangle.

This helps students understand that what makes a square a square is not its orientation on a page, but the fact that it is both a rhombus and a rectangle.

Other questions such as these will help students further their understanding of the properties that define each type of polygon.

“What quadrilaterals have opposite sides parallel?” (parallelograms, rectangles, rhombi, squares)

“What quadrilaterals have only one set of parallel sides?” (trapezoids)

“What shapes have opposite sides congruent? (parallelograms, rectangles, rhombi, squares)

“What shapes have adjacent sides congruent? (kites, squares, and rhombi)

It is suggested that students spend some time classifying quadrilaterals. This will help students who see a square and a rectangle as two unrelated shapes rather than seeing the square as a specific type of rectangle. It will also help them move away from identifying a square by its orientation on the page.

On the following pages are some diagrams that will help students in classifying quadrilaterals. We are moving to help students complete the chart in this fashion:

Two versions are provided. One has the names of the quadrilaterals and one has the characteristics. Once students have placed the polygons in the regions, they can remove the physical representations and replace them with a sketch.

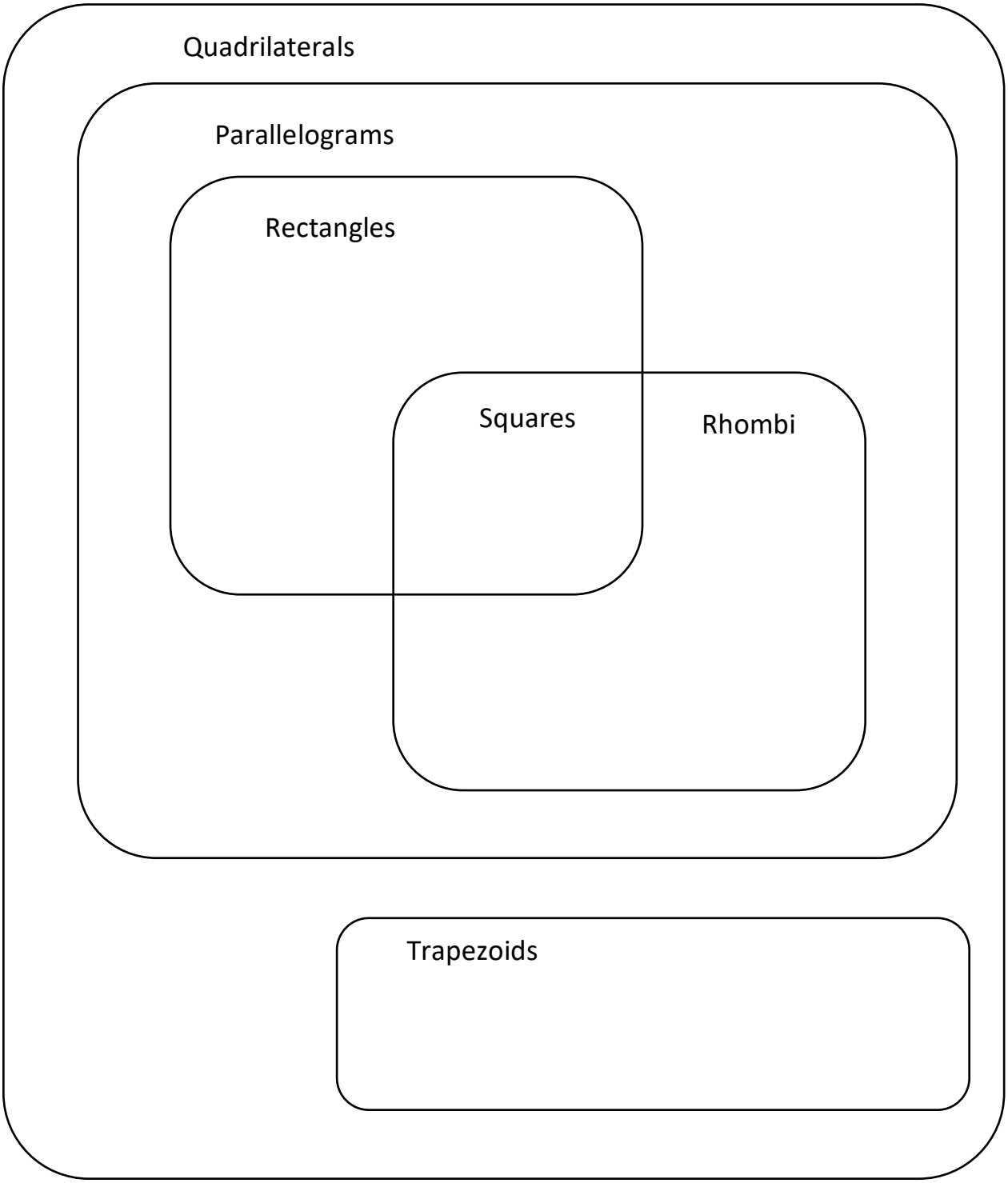
Following that is a similar activity that helps students classify triangles based on their properties. Again, two versions are provided.

Quadrilateral Classification

Name_____

Date_____ Class_____

Place attribute shapes in their correct regions. Then draw a sketch of each.

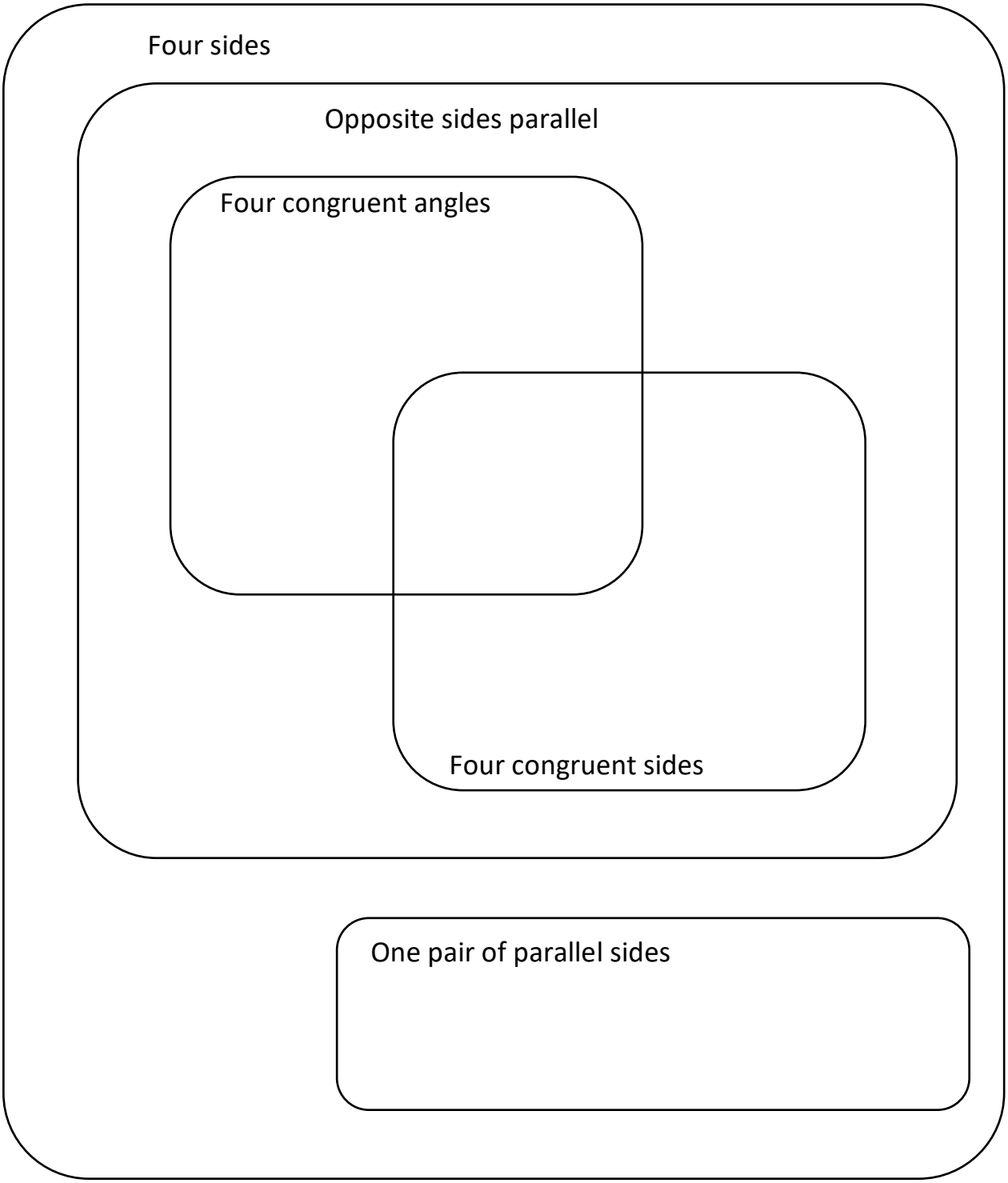


Quadrilateral Classification

Name_____

Date_____ Class_____

Place attribute shapes in their correct regions. Then draw a sketch of each.



Triangle Classification

Name_____

Date_____ Class_____

Place attribute shapes in their correct regions. Then draw a sketch of each.

	Scalene	Isosceles	Equilateral
Acute			
Right			
Obtuse			

Which regions do not have any triangles? _____

Why is that not possible? _____

_____

Triangle Classification

Name_____

Date_____ Class_____

Place attribute shapes in their correct regions. Then draw a sketch of each and write the name of the triangle

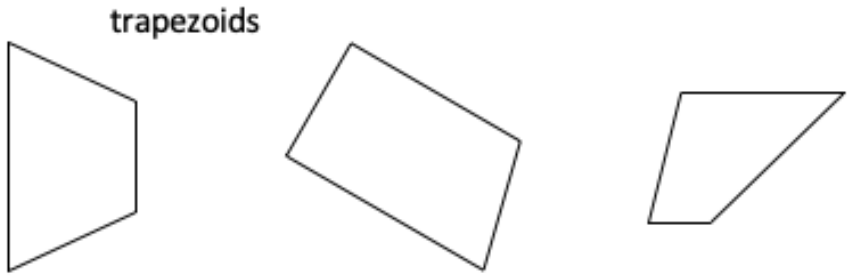
	No equal sides	Two equal sides	All 3 sides equal
All angles <90°			
One 90° angle			
One angle >90°			

Which regions do not have any triangles? _____

Why is that not possible? _____

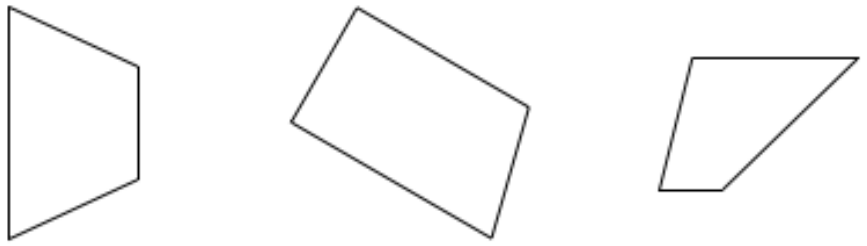
_____

Our next strategy takes students out of Visualization and into Analysis. I call it the “This is/This isn’t” approach. Here students are given examples and counterexamples. This is critical for this reason. Consider our last figure:

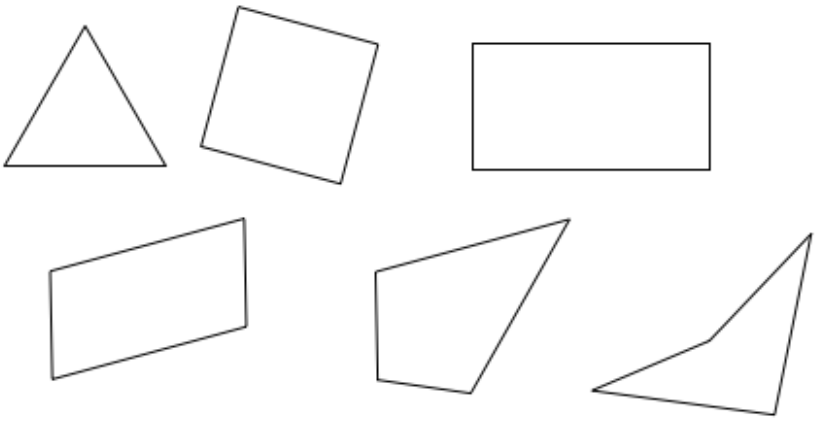


Students seeing these trapezoids for the first time may not notice that they have *one* pair of parallel sides. They may assume trapezoids are irregular quadrilaterals. In this case we can show the students:

These are trapezoids:



These are *not* trapezoids:

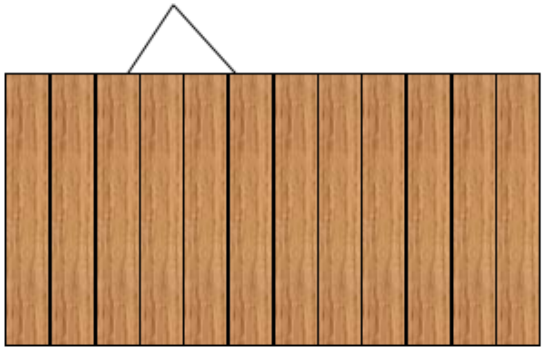


At this point, students are again forced to analyze the polygons to see what sets them apart. The counterexamples help students avoid the misconceptions mentioned above and set the stage for proof by negation later.

A good manipulative for this activity is a set of 2-D shape sets that can be obtained from a math manipulative supplier such as EAI. Here is a link to a set that I have found useful.

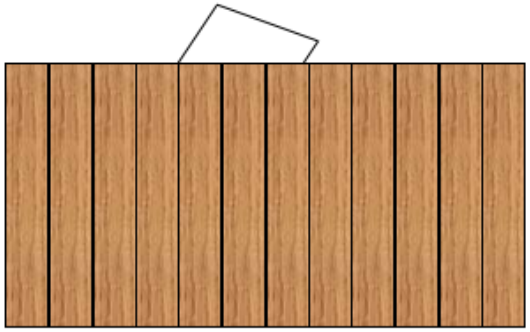
2-D Shape Set: www.eaieducation.com/Product/504937/2D_Shape_Sets.aspx

Another strategy is what I call “Peeking Polygons” (although this approach can be done with other concepts besides polygons). I show students *part* of a polygon peeking over a fence. I can choose to give them some starter clues or not. For example, if students see this, they don’t yet know if the polygon is a triangle, quadrilateral, or some polygon with more sides. I can then ask them, “Is there any shape that you can rule out based on what we see so far?” Since this is an acute angle, we know it is not a rectangle or square.

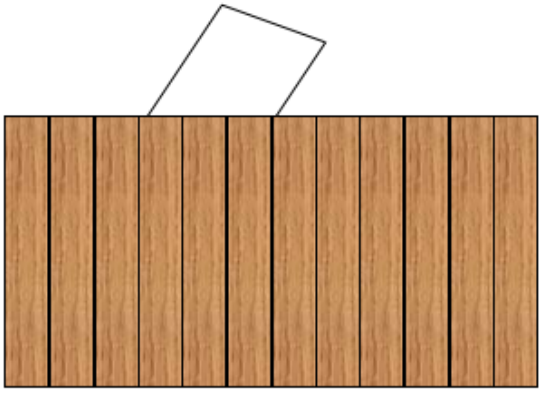


At this point, I can tell them that it is a quadrilateral if I wish. In that case, it can be a parallelogram, rhombus, trapezoid, or kite. It can also be an irregular quadrilateral. It might even be concave.

I then reveal a bit more of the polygon.



Now that we see a set of parallel sides, we know it is not irregular, but it might be a rhombus, parallelogram, or trapezoid. This third view rules out the rhombus since we can see that it doesn’t have four congruent sides.



Again, attribute shapes will be a helpful manipulative for this activity.

This activity works well on an overhead projector like we used in the “old days”. Your district likely has a warehouse full of them somewhere. A document camera will also work.

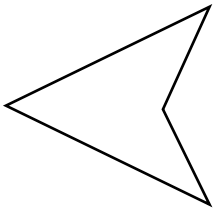
An overhead projector can also be used to project two-dimensional images of a three-dimensional shape. Construct a shield with folders so students don’t see the shape but only the projection of it. For example, placing a cylinder on the projector one way reveals a circular image, but laying it on its lateral surface reveals a rectangle. Students then try to guess what the shape is.

This brings us to another activity of using riddles to help students analyze the characteristics of a shape. We can give Geometry Riddles like this:

> "I am a quadrilateral."
>
> "I have two pairs of congruent sides."
>
> "I have no right angles."
>
> "I have no parallel sides."
>
> "One of my interior angles is over 180º."
>
> "Can you draw my picture and write my name?"

After the second clue, most students will be picturing a square, rectangle, rhombus, or parallelogram. However, the third clue tells us that it can't be a square or rectangle. The fourth clue seems to rule out all possible quadrilaterals. However, a concave kite fills the bill. The congruent sides are *adjacent*. This type of activity also reinforces vocabulary.

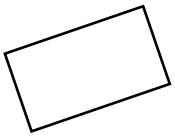
Here is a shape that satisfies the riddle above:



Some sample riddles are provided on the next page. These are for illustrative purposes. You will need to design clues appropriate to the skill level of you students.

Answers:

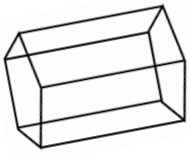
1 Rectangle



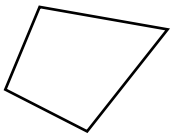
2 Regular hexagon

3 Square based pyramid

4 Pentagonal prism



5 Kite



Geometry Riddles:

1

“I am a quadrilateral.”

“My opposite sides are parallel.”

“I have at least one right angle.”

“Can you draw my picture and write my name?”

2

“I am an interior angle sum of 720º”.

“My sides are congruent.”

“Can you draw my picture and write my name?”

3

“I am a three-dimensional shape.”

“I have five sides.”

“I have five vertices.”

“How many edges do I have?”

“Can you draw my picture and write my name?”

4

“I am a three-dimensional shape.”

“I have seven surfaces.”

“I have two bases.”

“Can you draw my picture and write my name?”

5

“I am a quadrilateral.”

“Two of my angles are congruent.”

“My diagonals are perpendicular.”

“One of my diagonals bisects the other.”

“Can you draw my picture and write my name?”

Transitioning from Level 1 to Level 2: Abstraction

In moving from Analysis to Abstraction we are looking for activities that require students to analyze characteristics and properties and then to apply their observations across wider samples to see if the observations hold. This involves *inductive reasoning* – reasoning by observation – and is much like the work of a scientist. A scientist will notice a trait or characteristic. Then they will see if they can replicate the result. Other scientists will also try to replicate the findings. We are seeking to imitate this process in class.

For example, science doesn't *prove* something in the same way that a mathematician would. We can talk of the *law* of gravity, but sometimes we still refer to it as the gravitational *theory*. We have observed it over and over without exception, but we have not proven that gravity will always behave the same predictable way through the universe and the infinitude of time.

On the other hand, a student may observe that the sum of two odd numbers is always even. We can create countless examples:

$5 + 7 = 12$ $\quad$ $1 + 19 = 20$ $\quad$ $5 + 5 = 10$ $\quad$ $101 + 1{,}883 = 1{,}984$ $\quad$ $^{-}11 + 17 = 6$

However, there are still a lot of odd number pairs left that we haven't tried. We can assume that this observation will always hold true, but we can't be absolutely sure. It is not proven. It seems by noticing countless examples that all prime numbers are odd, but we find one counterexample (2) that disproves this.

With mathematics though, we can prove that two odd numbers add up to an even number. The odd numbers are defined by the formula $2n+1$ where n is any whole number. Thus, we can write the sum of two odd numbers in this manner:

$$(2n+1) + (2n+1) = 4n + 2$$

Since the formula for even numbers is $2n$, we can rewrite the sum as $2(2n) + 2$ which is clearly an even number.

This sort of proof is what we see in level 3: Deduction, but we are not there yet. For now we will be content to test our conjectures through observation and inductive reasoning.

Level 2: Abstraction – Interior Angle Sums Activity,

Lesson 1

For this activity, I use a set of fraction circle manipulatives. Again, I got mine from EAI, and you can use this link:

Fraction Circle Shapes:

www.eaieducation.com/Product/531352/QuietShape__Foam_Fraction_Circles__Blank_-_Set_of_51.aspx

I begin by asking the students how many degrees are in a full circle. In most elementary classes there is at least one skateboarder who knows this answer. If not, I tell them that we divide a circle into 360 degrees to measure angles. I show them the blue circle in the set as I explain this.

Then I take them through each color as we explore how many degrees each sector has. I make sure to use the term *sector* as I do this so that they know experientially that the definition of a sector is a section of a circle from the center to the perimeter.

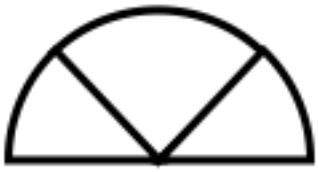
Once they realize that they are dividing 360º by the number of sectors to find the angle measure of each, I let them work in groups to finish the task card.

After this, I always refer to a sector by its angle measure instead of its color. Instead of saying "yellow", I say "120º sector."

Lesson 2

Now that students have established the angle measures of the different sectors, they are ready to use them as tools to explore polygons. Remember that with Abstraction, we are trying to help students **discover characteristics and properties and then test them across multiple examples** to see if there are exceptions.

Students measure the interior angles of the first triangle using their sectors. They see that angle A measures 45º, angle B measures 90º, and angle C measures 45º. Then they take these three sectors and lay them so that their vertices meet as shown:



Now they see that the interior angle sum of triangle 1 is 180º. They proceed in a similar manner as they measure the interior angles of triangles 2, 3, and 4, and they notice that in all four cases, the interior angle sum remains 180º. They then use the same process for the quadrilaterals, the pentagons, and the hexagons. They observe that the interior angle sum increases by 180º each time an additional side is added.

Notice what we have asked the students to do. They made an observation about the sum of the interior angles of a triangle. Then they tested this on other triangles. After it worked four times, they began to *generalize* this observation. That is the purpose of the Abstraction level: to make conjectures about geometry. These conjectures will later be proven during level 3: Deduction.

A logical extension of this activity would be to ask the students these questions:

> "Do you think that *all* triangles have an interior angle sum of 180º?"
>
> "Is it possible that it only works with the common angle measures of our sectors?"
>
> "Do you think it would work with random triangles with less common angle measures?"

Have students construct their own random triangles using a straightedge. Then they can use a protractor to measure the angles and find the sum.

I once started this lesson this way. Each student drew a triangle. Then we talked about the multitude of triangles that they had created. There were acute, obtuse, and right triangles. There were equilateral, isosceles, and scalene triangles. There were small triangles and big ones. Then I had them find the measures of their three angles and add them. "On the count of three, I want you to shout out what your angle sum is," I proudly instructed. "One…two…three!"

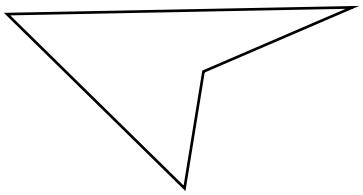
Students shouted, "180º!", "179º!", "181º!", "257º!" Since they had no reference, their inaccuracies in measurement didn't alarm them. Many students used the wrong scale on their protractor and were not bothered by the discrepancy. They didn't know their answers should match.

By using the fraction circle sectors, measurement errors are virtually eliminated. Since they already suspect the angle sum will be 180º, those students who measure 182º with their protractors will likely go back to check their work. They will try to make their triangle obey the rules.

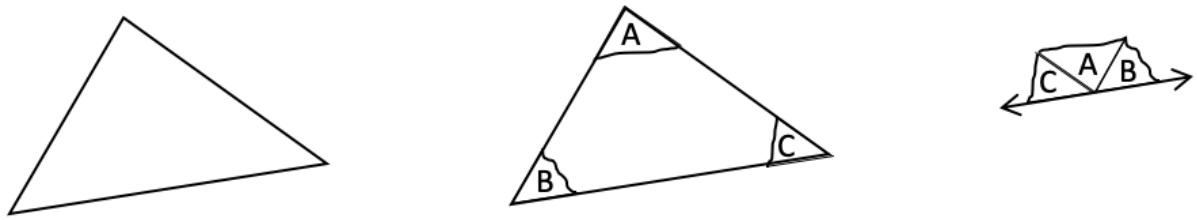
I ask the students to compare the interior angle sum of their triangle with the rest of their group. If three students get 180º and one gets 257º, then that student realizes that they have probably used the wrong scale on their protractor.

Then I have the groups report their results and they see that *every* group and *every* student got an interior angle sum of 180º on their randomly drawn triangle.

This can be extended into quadrilaterals, pentagons, and other polygons. Encourage students to push the limits. Does it work for an irregular, concave quadrilateral?

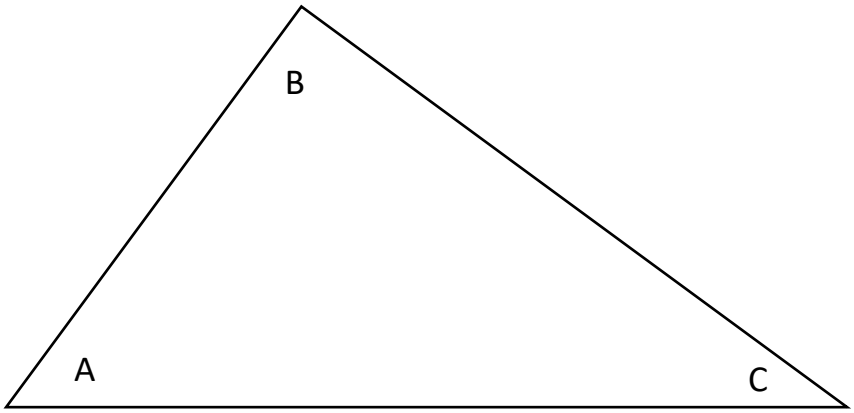


Another way to demonstrate this is with the activity mentioned previously in which the students tear the corners off of a triangle and lay their vertices together.

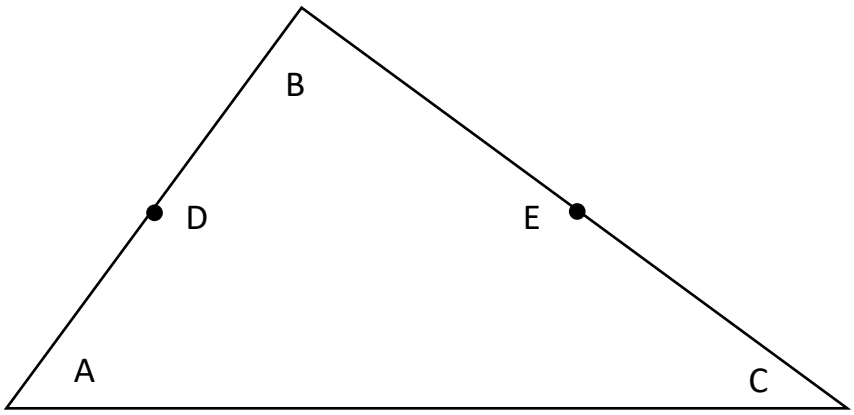


As they compare their results with those of other students they begin the generalization that is critical to the Abstraction level.

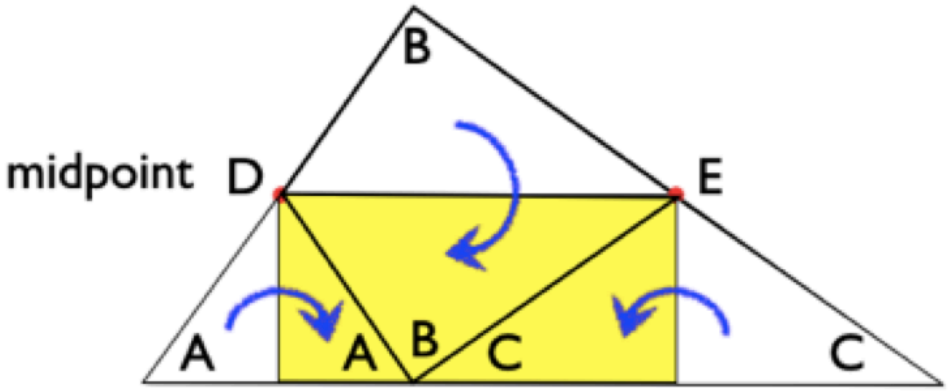
Here is another very clever illustration that requires no formal skill in geometry construction. Ask the students to construct a triangle similar to the one shown. Angle A and angle C must be acute angles for this to work.



Label the vertices A, B, and C as shown. Find the midpoint of side AB. This can be done with compass and straightedge if your students are ready for that, but the midpoint can also be found by accurately folding vertex A to B. Label this point D. Find the midpoint of side BC the same way and label it point E. Construct segment DE, the median of the triangle.



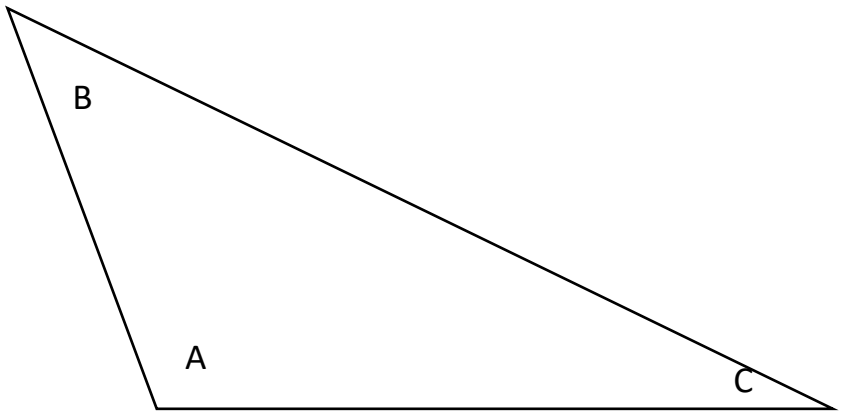
Now fold down on segment DE. Point B will touch base AC. Fold vertex A to meet at B. Fold vertex C to meet at B as well.



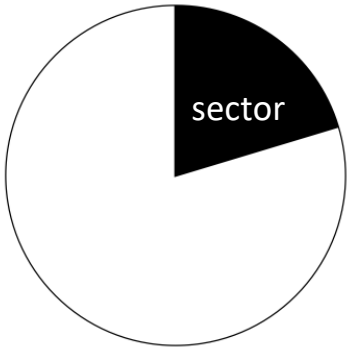
What is the sum of angles A + B + C? (180º)

What is the area of the rectangle compared to the area of triangle ABC? (It is half the area.)

What happens if we try this with the tringle below? (Folding along the median puts vertex B outside of the triangle making the observation more difficult to see.



1. Angle Measures



There are _____ degrees in a full circle. Use this information to find the measures of each angle sector.

Color	Number of Pieces	Angle measure
Blue		
Brown		
Black		
Yellow		
Green		
Orange		
Red		
Purple		
Pink		

Now use this information to answer the questions on the back.

Write your answers to each question using the angle measures of the sectors instead of the colors. Don't forget to label your measurements using the degree symbol.

1. Yellow + Orange = _____ + _____ = _____
2. Purple + Green = _____ + _____ = _____
3. Make 75º with two pieces _____ + _____ = _____
4. Make 90º using two different colors _____ + _____ = _____
5. Make 81º _____ + _____ = _____
6. Make 138º _____
7. Show two ways of making 135º

   _____ _____

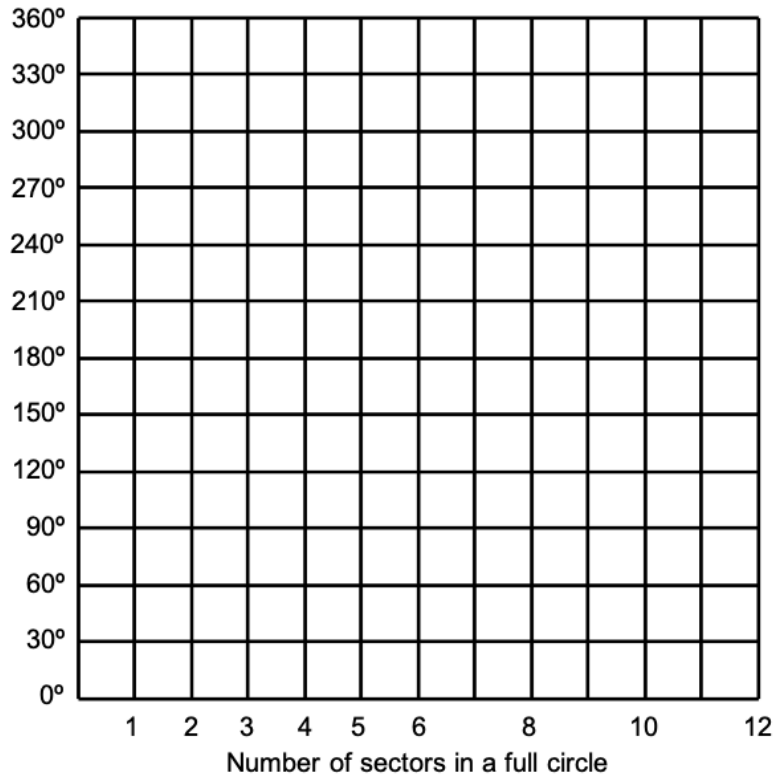
8. Set a yellow sector on top of black sector, how many degrees of the black are showing?

   Black – Yellow = _____ – _____ = _____

9. Make a circle using the most colors. Record your answer here.

   _____

10. Graph the degrees of each sector.



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3. Make 75º with two pieces _____ + _____ = _____
4. Make 90º using two different colors _____ + _____ = _____
5. Make 81º _____ + _____ = _____
6. Make 138º _____
7. Show two ways of making 135º

   _____ _____

8. Set a yellow sector on top of black sector, how many degrees of the black are showing?

   Black – Yellow = _____ – _____ = _____

9. Make a circle using the most colors. Record your answer here.

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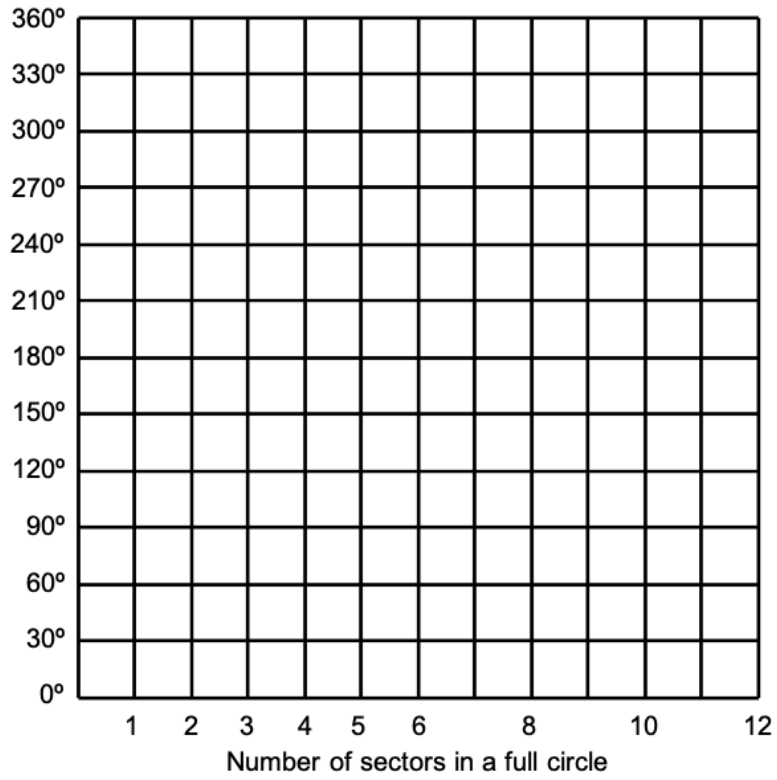
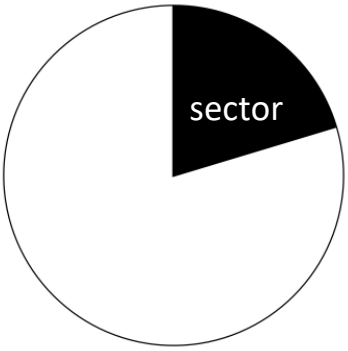


1. Angle Measures



There are **360** degrees in a full circle. Use this information to find the measures of each angle sector.

Color	Number of Pieces	Angle measure
Blue	**1**	**360º**
Brown	**2**	**180º**
Black	**3**	**120º**
Yellow	**4**	**90º**
Green	**5**	**72º**
Orange	**6**	**60º**
Red	**8**	**45º**
Purple	**10**	**36º**
Pink	**12**	**30º**

Now use this information to answer the questions on the back.

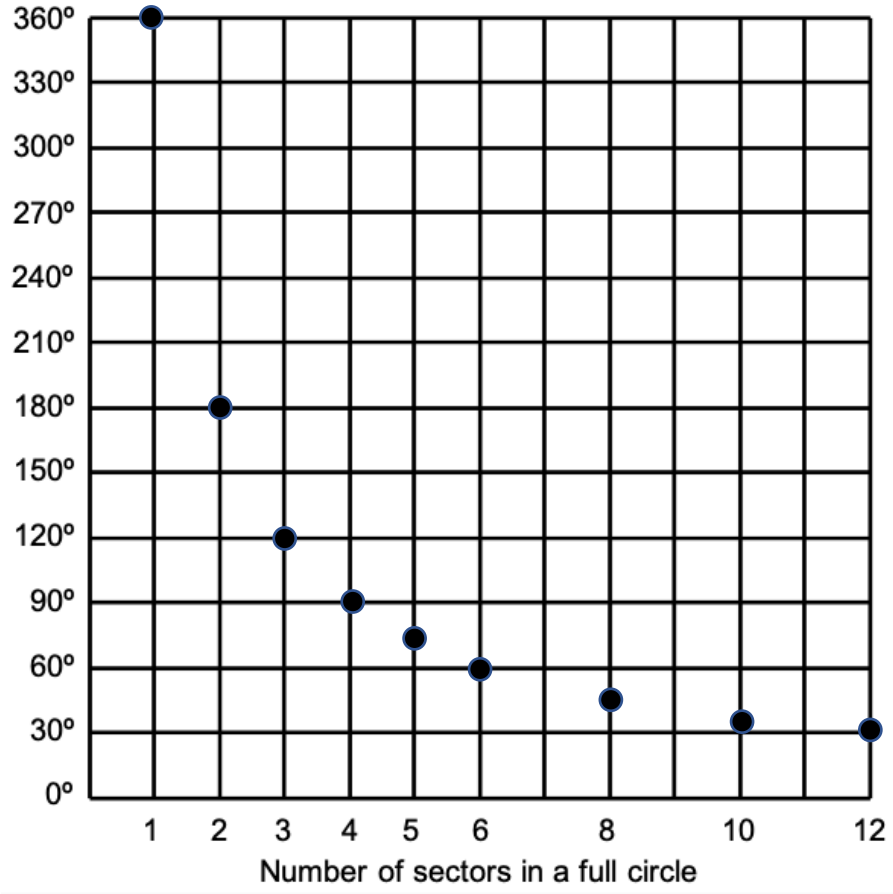
Write your answers to each question using the angle measures instead of the colors. Don't forget to label your measurements using the degree symbol.

1. Yellow + Orange = 90º + 60º = 150º
2. Purple + Green = 36º + 72º = 108º
3. Make 75º with two pieces. 30º + 45º = 75º
4. Make 90º using two different colors. 30º + 60º = 90º
5. Make 81º. 36º + 45º = 81º
6. Make 138º. 30 + 36º + 36º + 36º
7. Show two ways of making 135º.

   45º + 90º 45º + 45º + 45º 30º + 45º + 60º 30º + 30º + 30º + 45º
8. Set a yellow sector on top of black sector, how many degrees of the black are showing?

   Black – Yellow = 120º – 90º = 30º
9. Make a circle using the most colors. Record your answer here.

   36º + 36º + 36º + 60º + 30º + 72º + 90º (Five colors, seven pieces)
10. Graph the degrees of each sector.



2 Interior Angles

Use your Fraction Circle sectors to measure the angles of each polygon and find the interior angle sum.

Polygon	Angle A	B	C	D	E	F	Sum
Triangle 1	_____	_____	_____				_____
Triangle 2	_____	_____	_____				_____
Triangle 3	_____	_____	_____				_____
Triangle 4	_____	_____	_____				_____
Quadrilateral 1	_____	_____	_____	_____			_____
Quadrilateral 2	_____	_____	_____	_____			_____
Quadrilateral 3	_____	_____	_____	_____			_____
Quadrilateral 4	_____	_____	_____	_____			_____
Pentagon 1	_____	_____	_____	_____	_____		_____
Pentagon 2	_____	_____	_____	_____	_____		_____
Hexagon 1	_____	_____	_____	_____	_____	_____	_____
Hexagon 2	_____	_____	_____	_____	_____	_____	_____

Heptagon (7 sides) _____

What patterns did you notice?

_____

_____

Can you find a formula for the interior angle sum of a polygon where *n* is the number of sides? _____

Icosagon (20 sides) _____

2 Interior Angles

Use your Fraction Circle sectors to measure the angles of each polygon and find the interior angle sum.

Polygon	Angle A	B	C	D	E	F	Sum
Triangle 1	_____	_____	_____				_____
Triangle 2	_____	_____	_____				_____
Triangle 3	_____	_____	_____				_____
Triangle 4	_____	_____	_____				_____
Quadrilateral 1	_____	_____	_____	_____			_____
Quadrilateral 2	_____	_____	_____	_____			_____
Quadrilateral 3	_____	_____	_____	_____			_____
Quadrilateral 4	_____	_____	_____	_____			_____
Pentagon 1	_____	_____	_____	_____	_____		_____
Pentagon 2	_____	_____	_____	_____	_____		_____
Hexagon 1	_____	_____	_____	_____	_____	_____	_____
Hexagon 2	_____	_____	_____	_____	_____	_____	_____

Heptagon (7 sides) _____

What patterns did you notice?

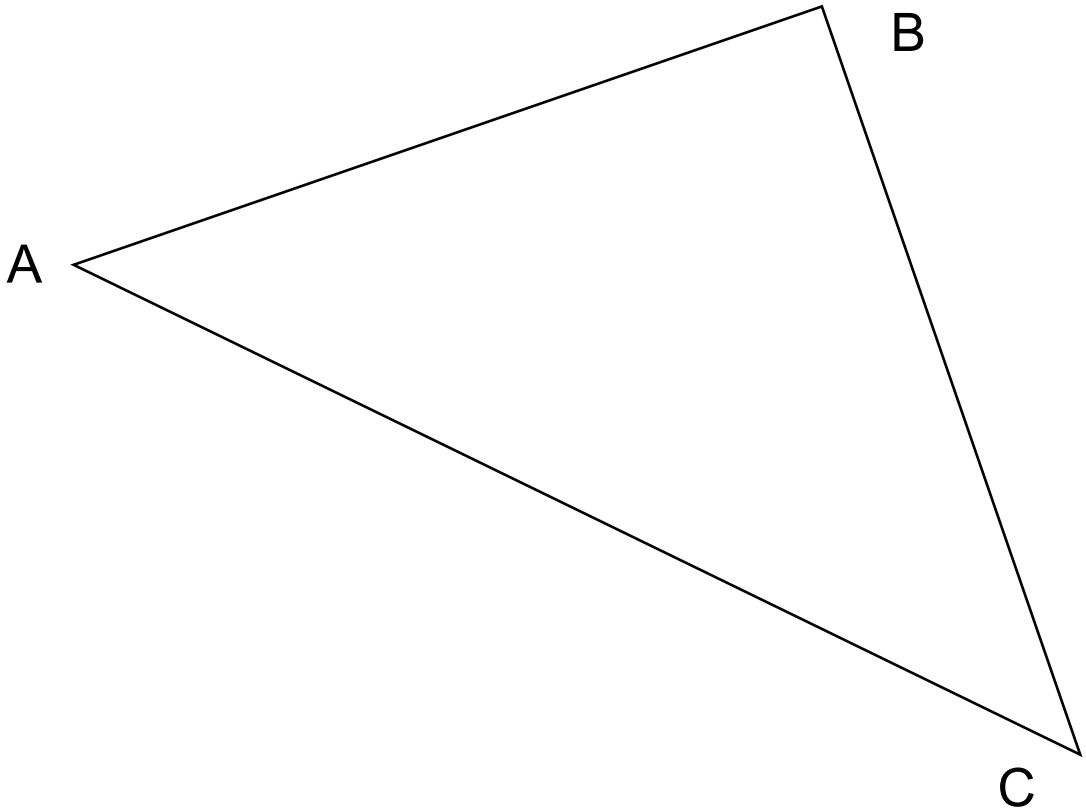
_____

_____

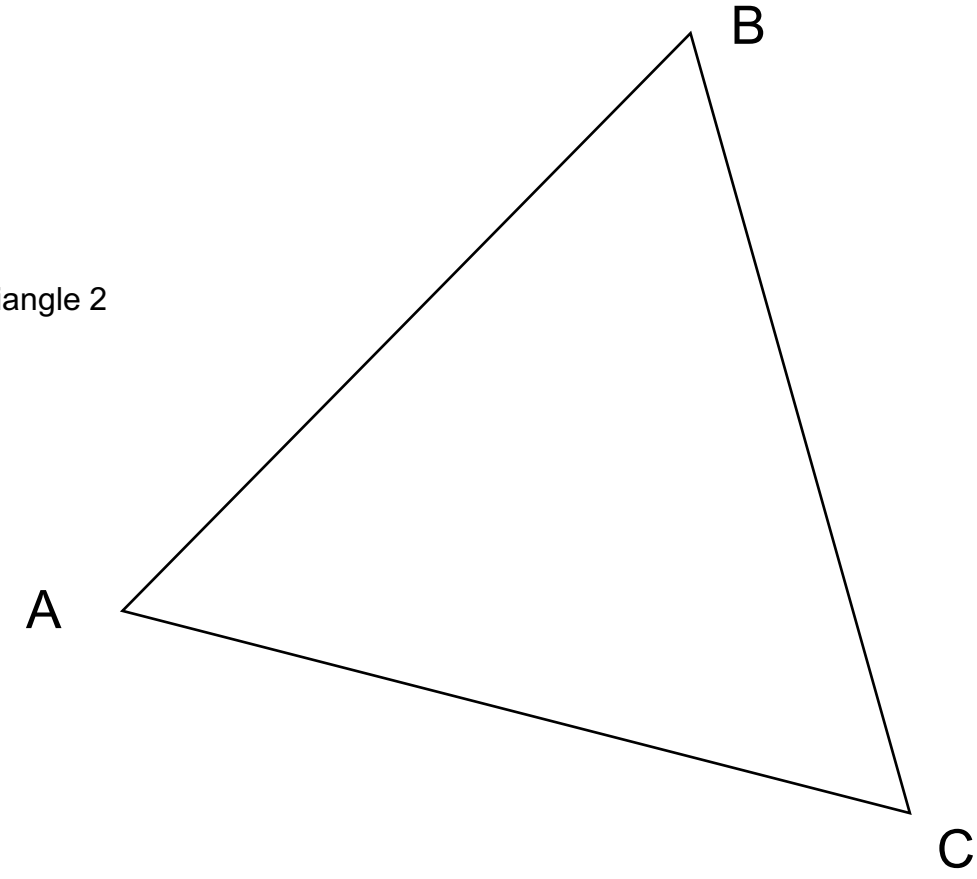
Can you find a formula for the interior angle sum of a polygon where *n* is the number of sides? _____

Icosagon (20 sides) _____

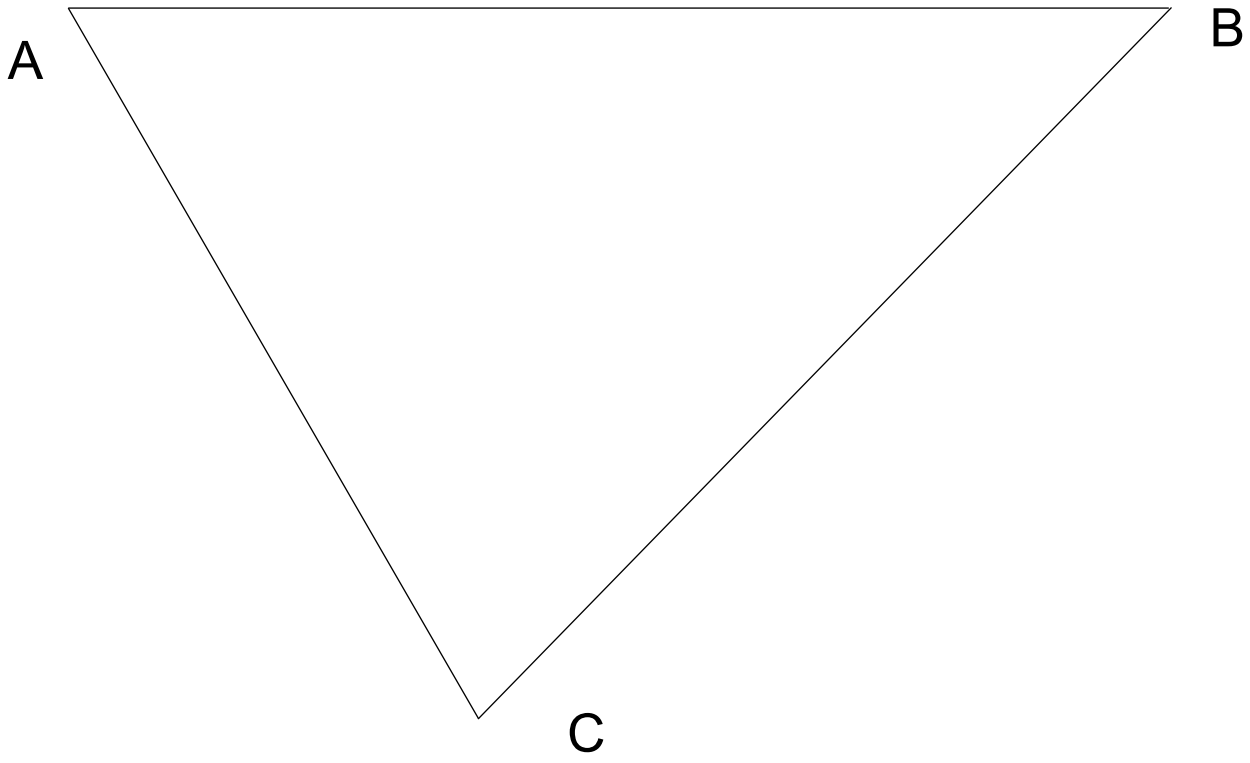
Triangle 1



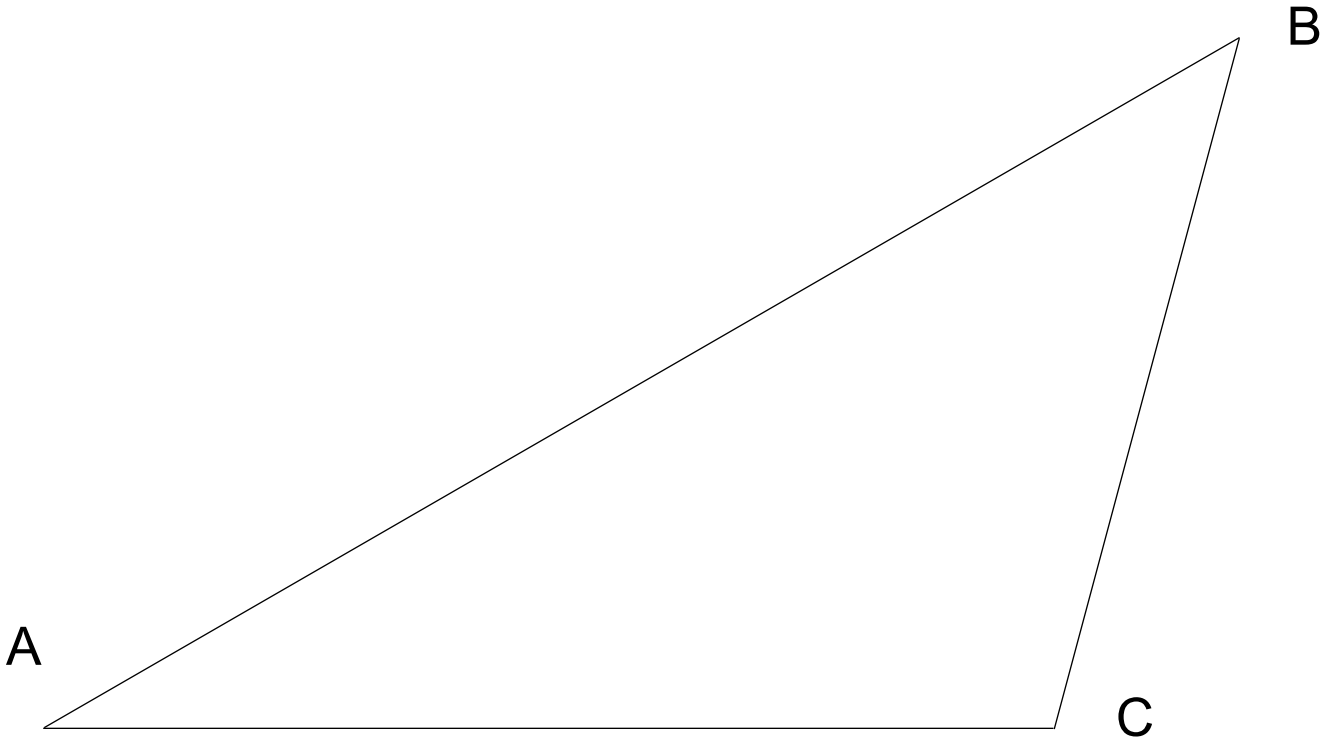
Triangle 2



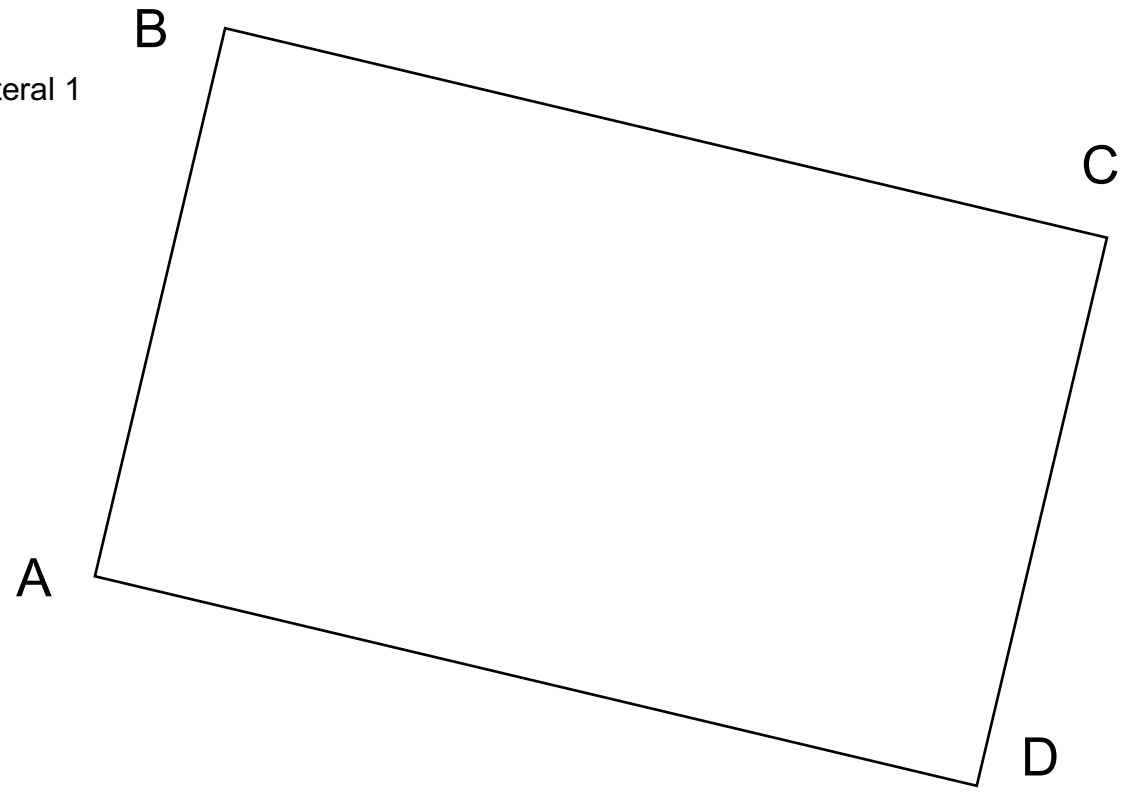
Triangle 3



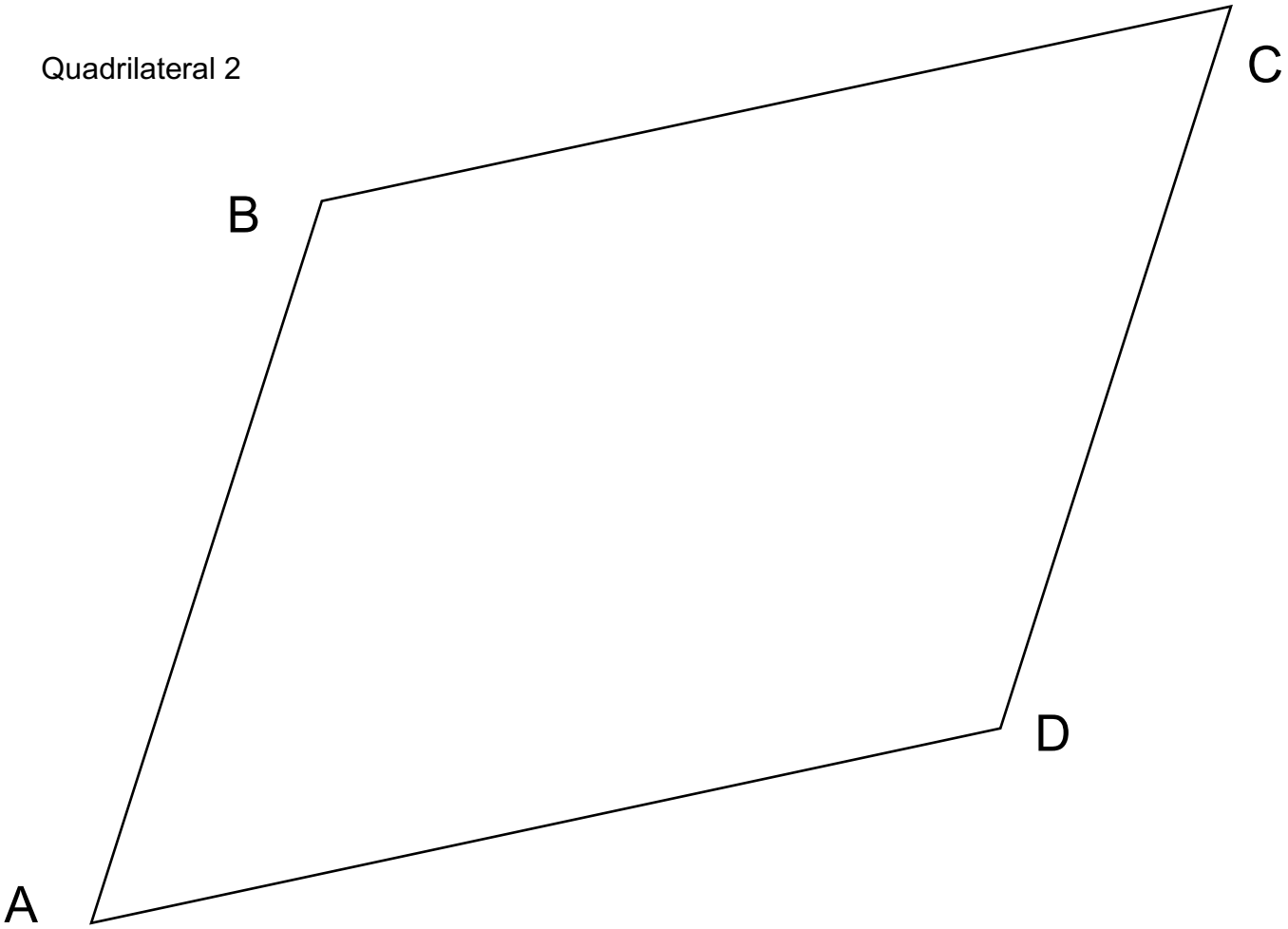
Triangle 4



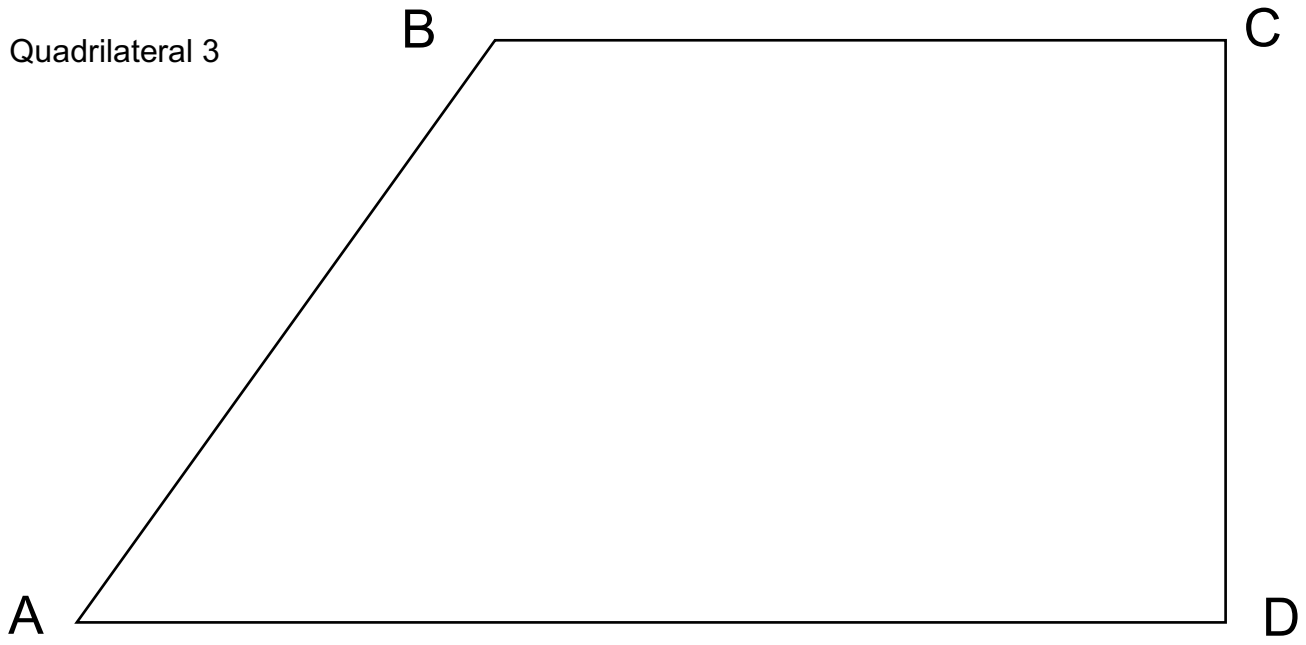
Quadrilateral 1



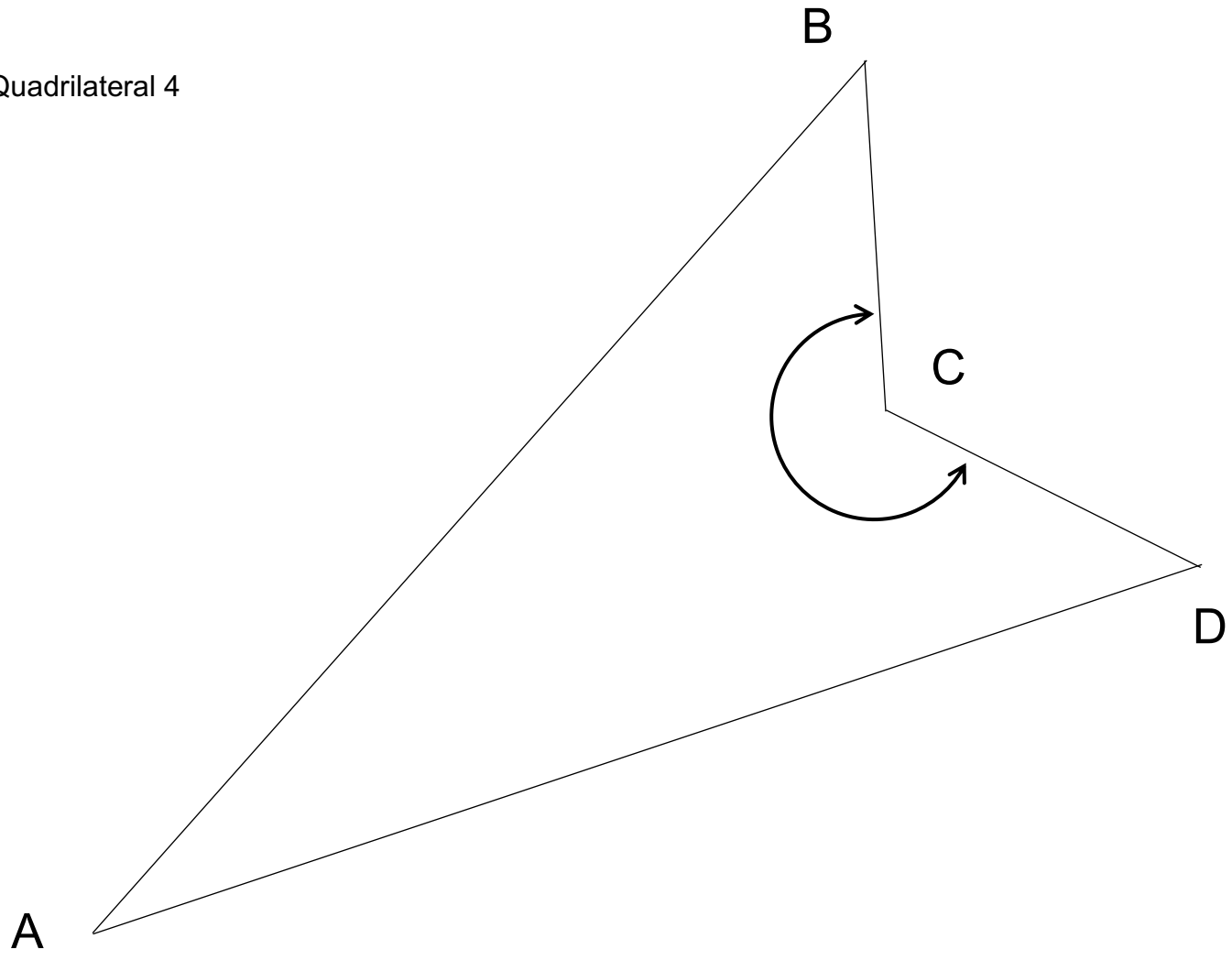
Quadrilateral 2



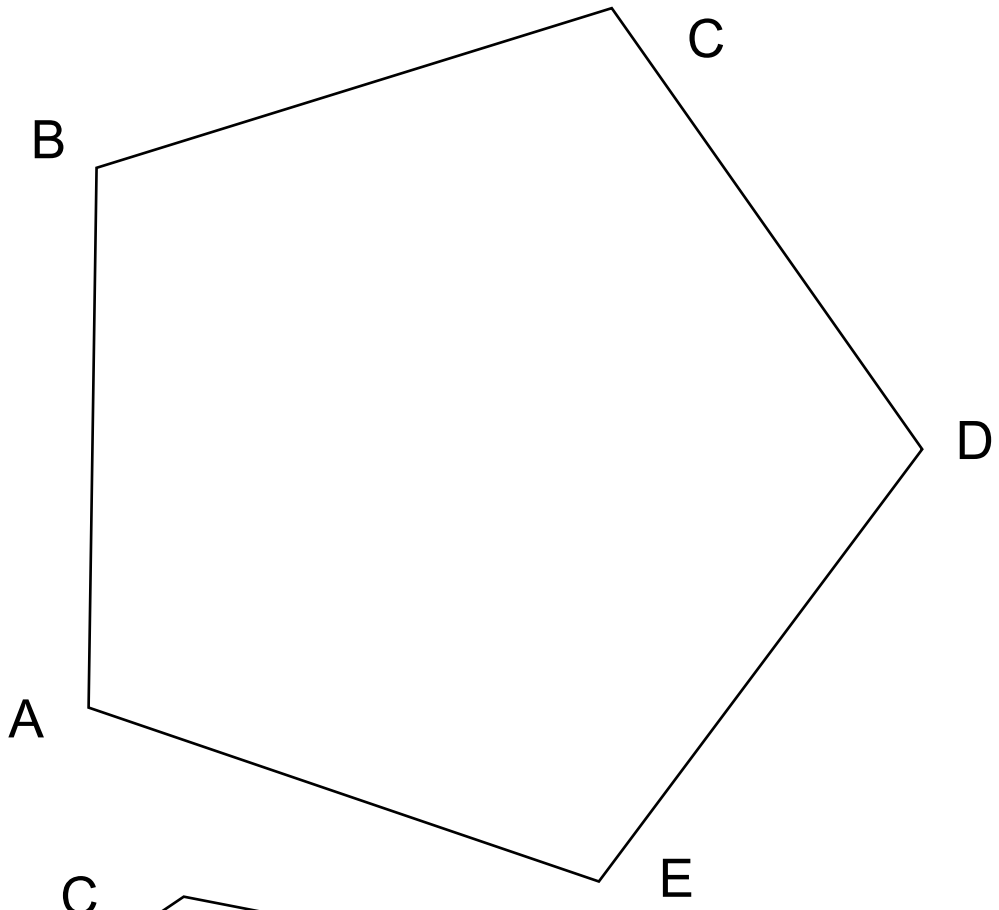
Quadrilateral 3



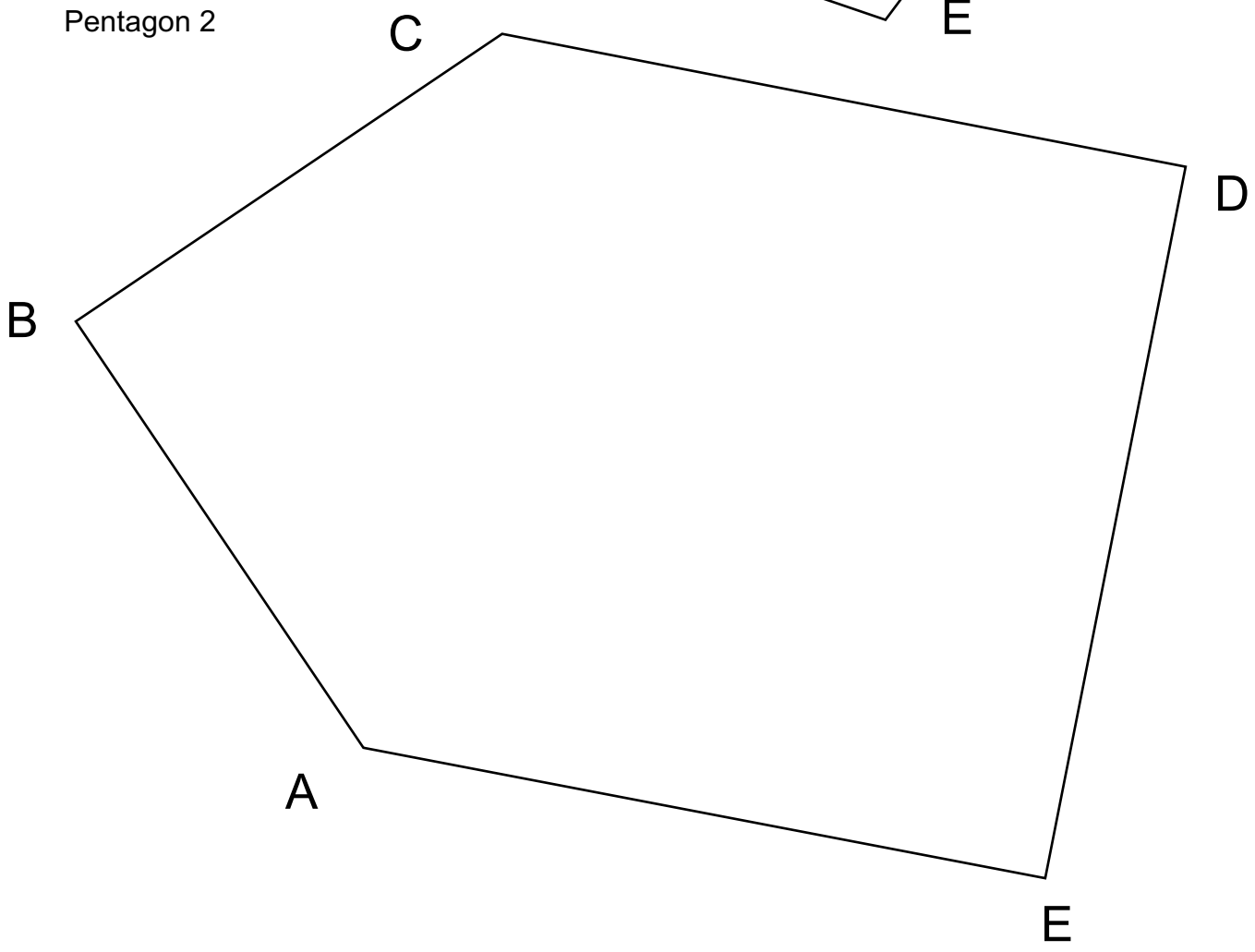
Quadrilateral 4



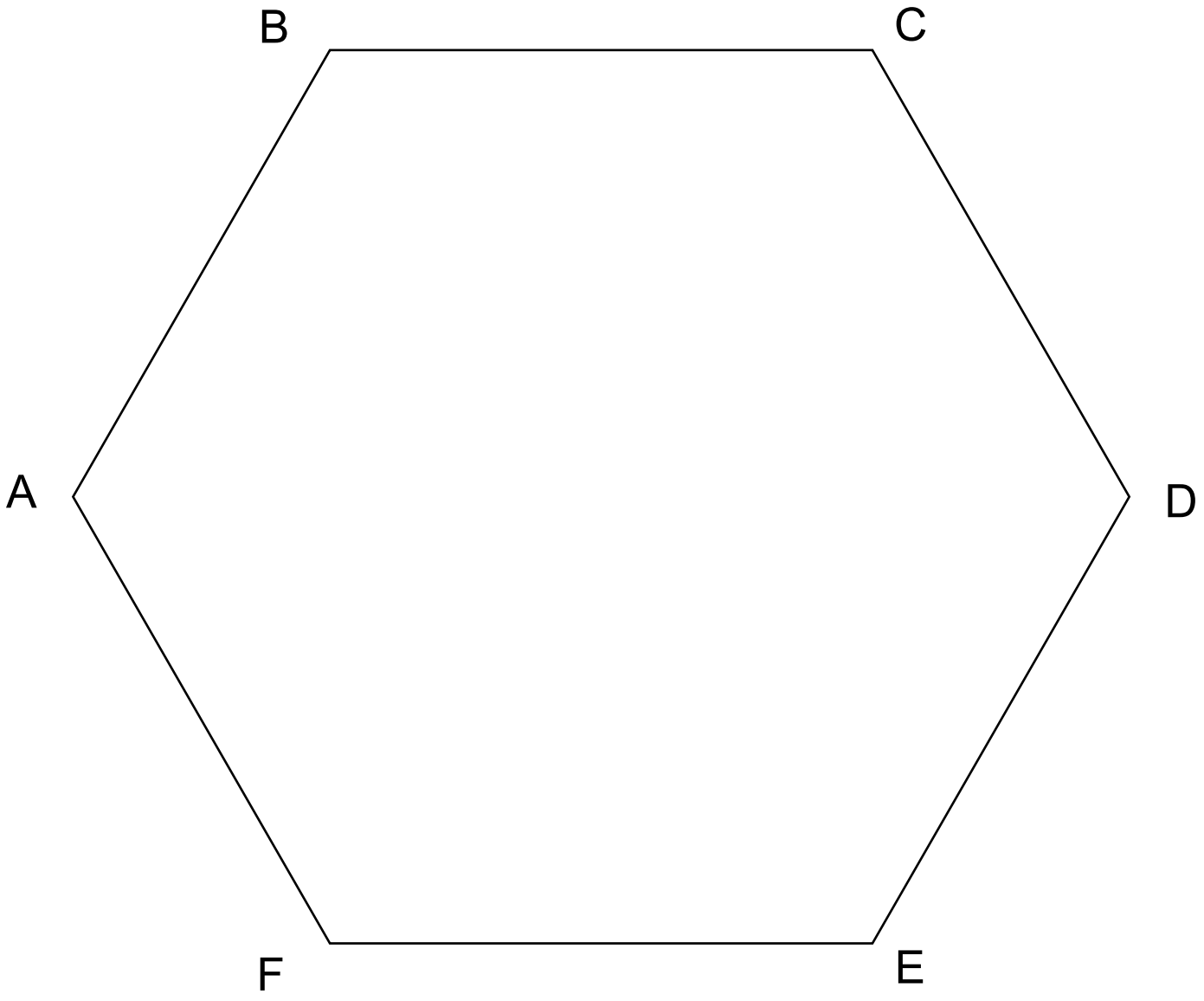
Pentagon 1



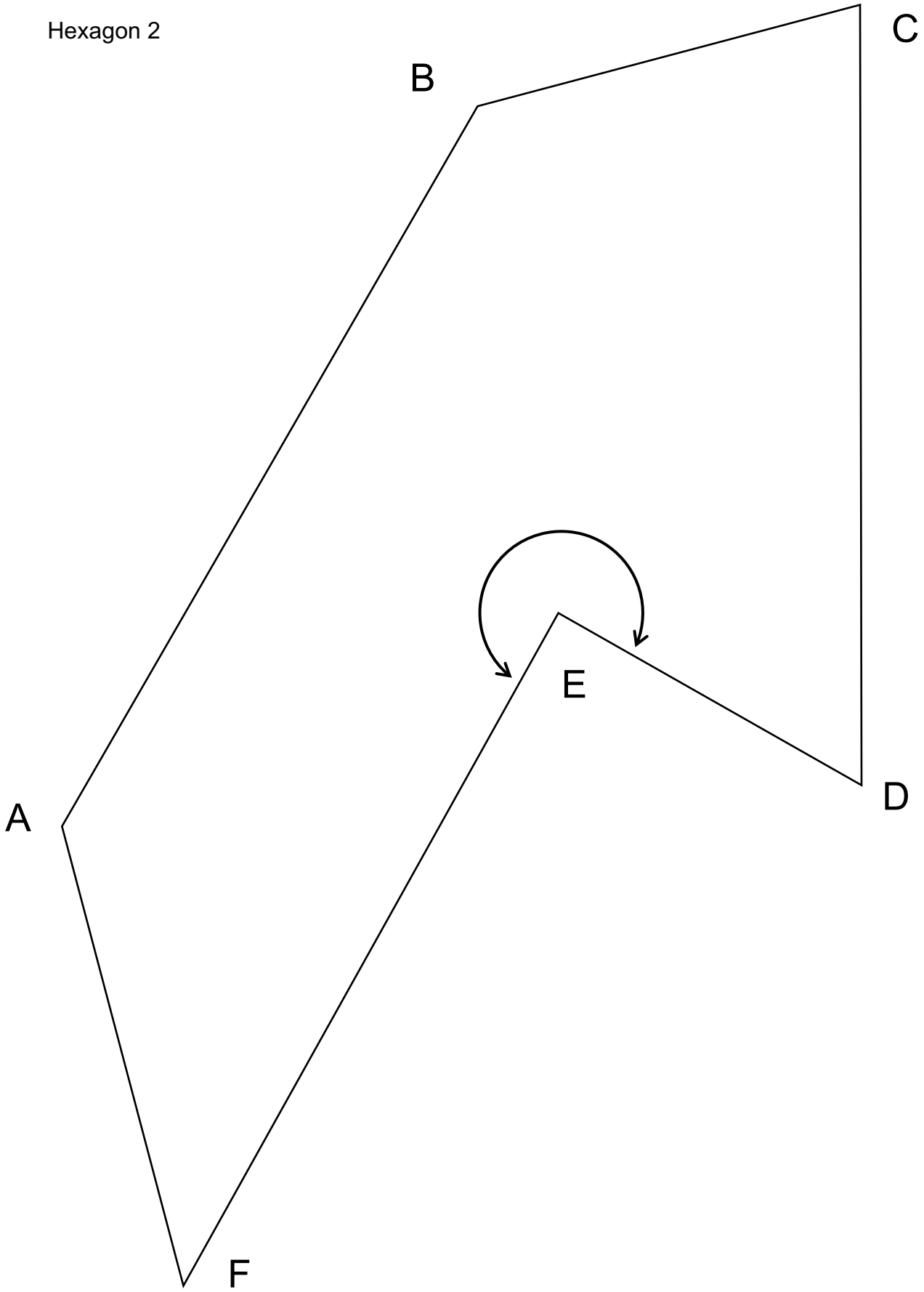
Pentagon 2



Hexagon 1



Hexagon 2



2 Interior Angles

Use your Fraction Circle sectors to measure the angles of each polygon and find the interior angle sum.

Polygon	Angle A	B	C	D	E	F	Sum
Triangle 1	45º	90º	45º				180º
Triangle 2	60º	60º	60º				180º
Triangle 3	60º	45º	75º				180º
Triangle 4	30º	45º	105º				180º
Quadrilateral 1	90º	90º	90º	90º			360º
Quadrilateral 2	60º	120º	60º	120º			360º
Quadrilateral 3	60º	120º	90º	90º			360º
Quadrilateral 4	30º	45º	240º	45º			360º
Pentagon 1	108º	108º	108º	108º	108º		540º
Pentagon 2	135º	90º	135º	90º	90º		540º
Hexagon 1	120º	120º	120º	120º	120º	120º	720º
Hexagon 2	135º	135º	75º	60º	270º	45º	720º
Heptagon (7 sides)							900º

What patterns did you notice?

Each time that a new side is added, the interior angle sum increases by 180º.

Can you find a formula for the interior angle sum of a polygon where *n* is the number of sides? $S = 180(n - 2)$ or $S = 180n - 360$

Icosagon (20 sides) 3,240º

A logical extension of this activity is to have students go back and measure the *exterior* angle sums of polygons. They will notice that regardless of the number of sides, the exterior angle sum is always 360º. They can then construct random polygons and measure the exterior angle sum with a protractor to verify that it seems to work for all tested polygons.

In the following pages, the students use their sectors to measure the exterior angles of triangles, quadrilaterals, pentagons, and hexagons. Their work is recorded on the activity sheet.

Exterior Angle Sums

Name_____

Date_____ Class_____

Use the sectors to measure the angles of the polygons and record your work on these pages.

Angle	A	B	C	D	E	F	Sum
Triangle 1	_____	_____	_____				_____
Triangle 2	_____	_____	_____				_____
Triangle 3	_____	_____	_____				_____
Triangle 4	_____	_____	_____				_____
Quadrilateral 1	_____	_____	_____	_____			_____
Quadrilateral 2	_____	_____	_____	_____			_____
Quadrilateral 3	_____	_____	_____	_____			_____
Quadrilateral 4	_____	_____	_____	_____			_____
Pentagon 1	_____	_____	_____	_____	_____		_____
Pentagon 2	_____	_____	_____	_____	_____		_____
Hexagon 1	_____	_____	_____	_____	_____	_____	_____
Hexagon 2	_____	_____	_____	_____	_____	_____	_____

Previously you calculated the interior angle sums for various polygons. Fill in your data from that activity in the first column. Then fill in the data for the exterior angle sums for each type of polygon using the following activity pages.

Polygon:	Interior Angle Sum	Exterior Angle Sum
Triangles	_____	_____
Quadrilaterals	_____	_____
Pentagons	_____	_____
Hexagons	_____	_____
…		
Decagons (ten sides)	_____	_____

A polygon has an interior angle sum of 1800º. How many sides does it have?

_____

Is it possible to make a 2-sided polygon? _____

If you could, what would be the interior angle sum? _____

Exterior Angle Sums Answer Key

Angle	A	B	C	D	E	F
Triangle 1	135º	135º	90º			
Triangle 2	120º	120º	120º			
Triangle 3	135º	105º	120º			
Triangle 4	135º	75º	150º			
Quadrilateral 1	90º	90º	90º	90º		
Quadrilateral 2	120º	60º	120º	60º		
Quadrilateral 3	90º	90º	135º	45º		
Quadrilateral 4	150º	135º	**-60º**	135º		
Pentagon 1	72º	72º	72º	72º	72º	
Pentagon 2	45º	90º	45º	90º	90º	
Hexagon 1	60º	60º	60º	60º	60º	60º
Hexagon 2	105º	120º	**-90º**	135º	45º	45º

Polygon:	Interior Angle Sum	Exterior Angle Sum
Triangles	180º	360º
Quadrilaterals	360º	360º
Pentagons	540º	360º
Hexagons	720º	360º
…		
Decagons (ten sides)	1,440º	360º

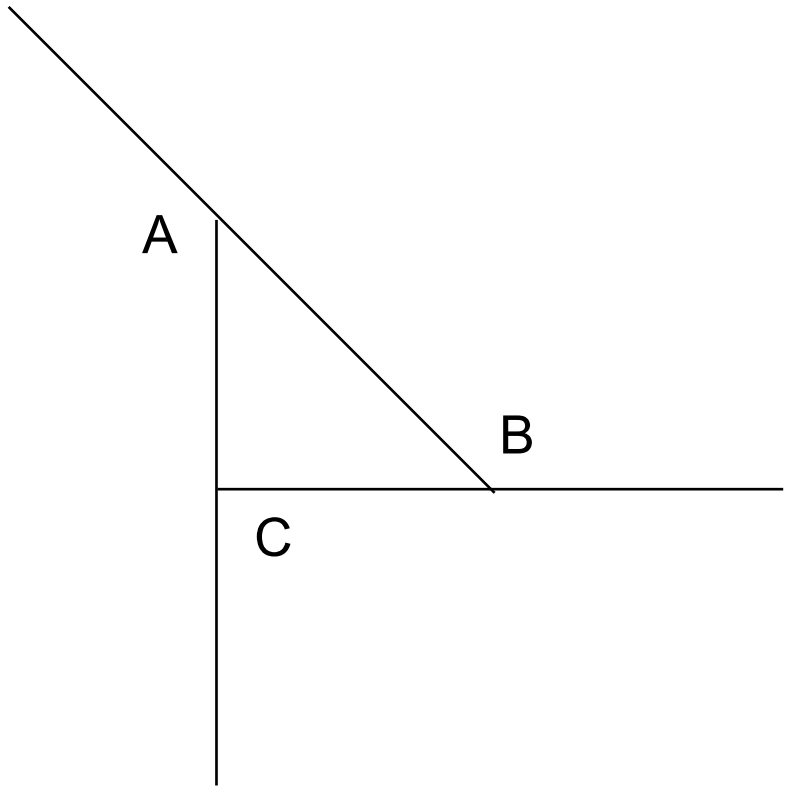
A polygon has an interior angle sum of 1800º. How many sides does it have?

12, it is a dodecagon

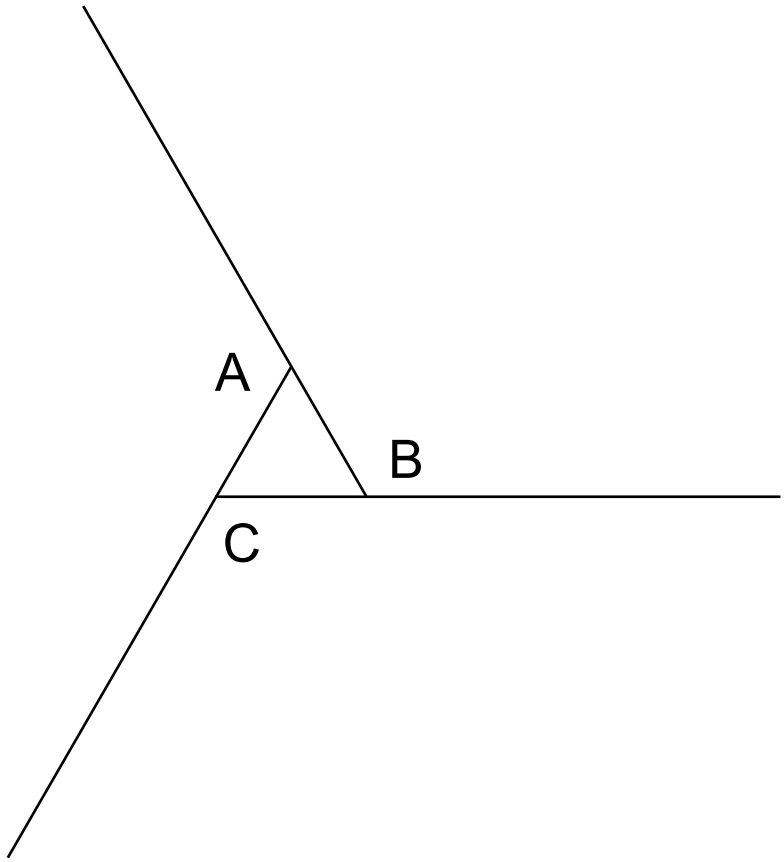
Is it possible to make a 2-sided polygon? No, it has no interior.

If you could, what would be the interior angle sum? 0º

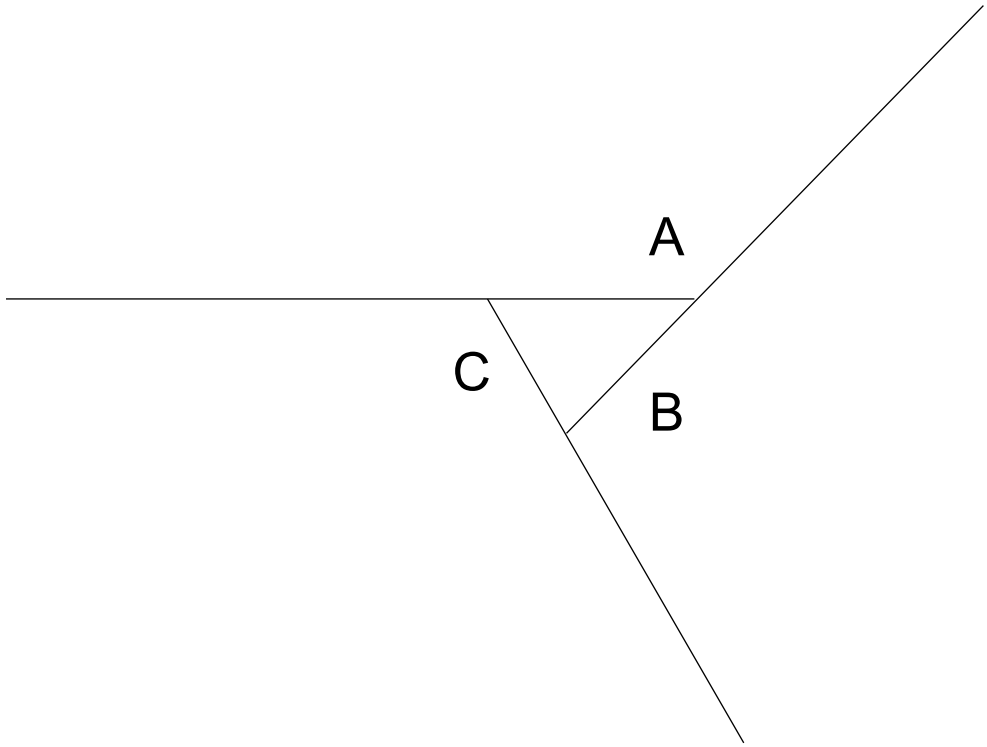
Triangle 1



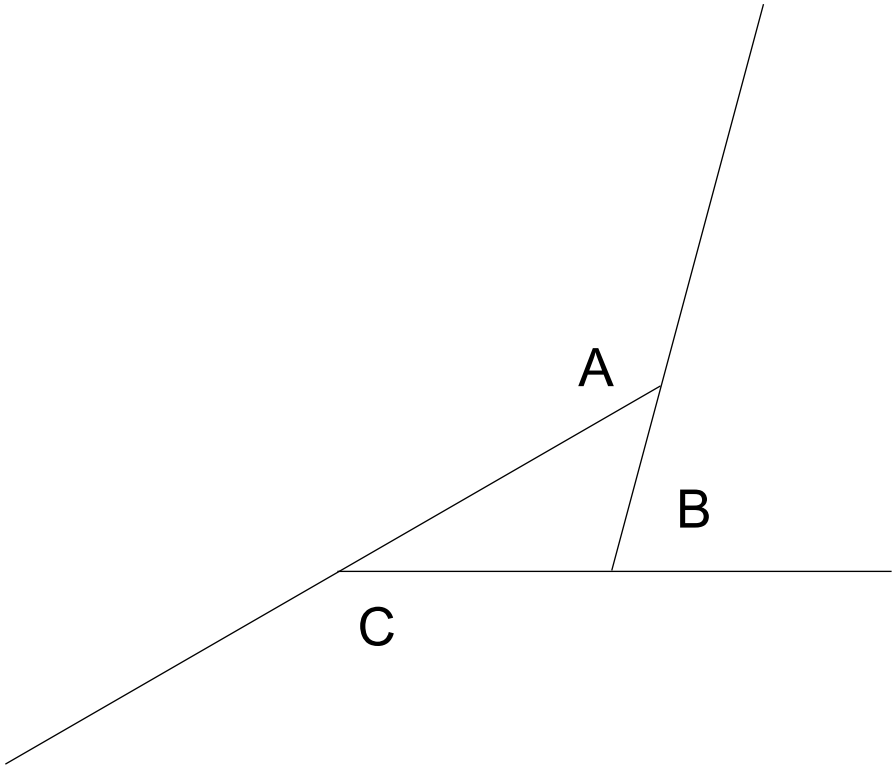
Triangle 2



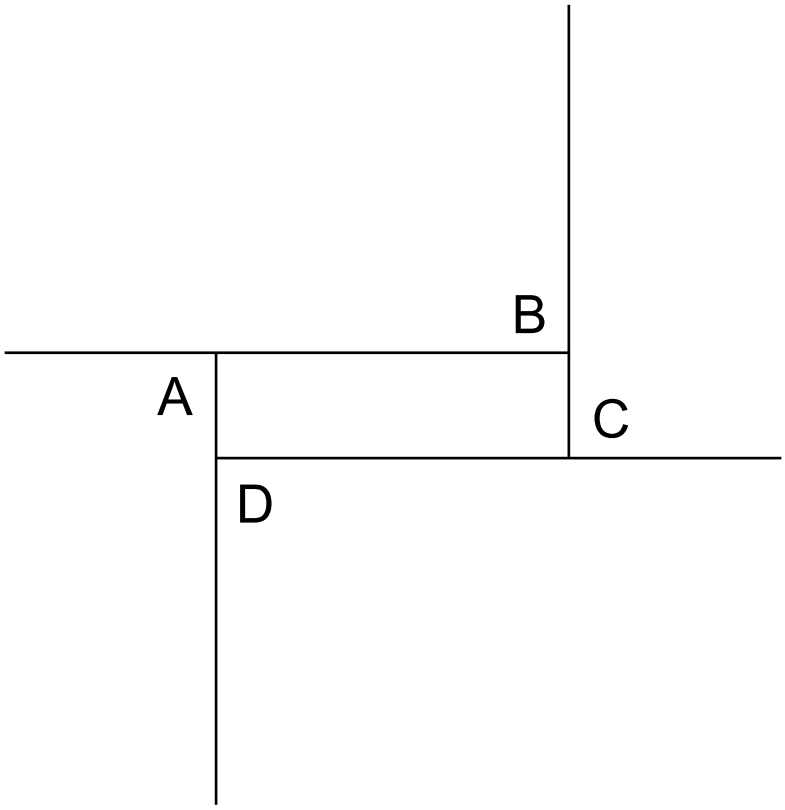
Triangle 3



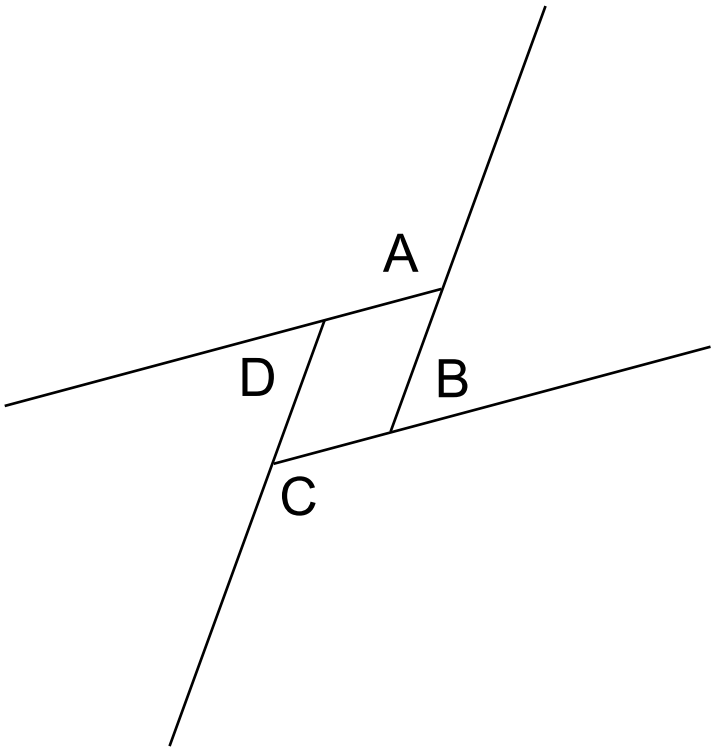
Triangle 4



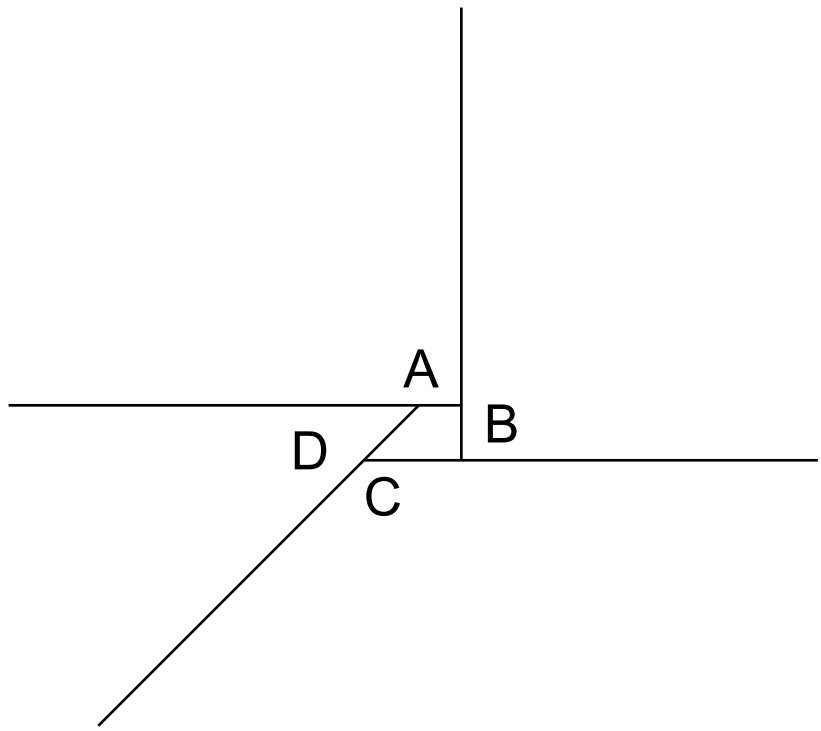
Quadrilateral 1



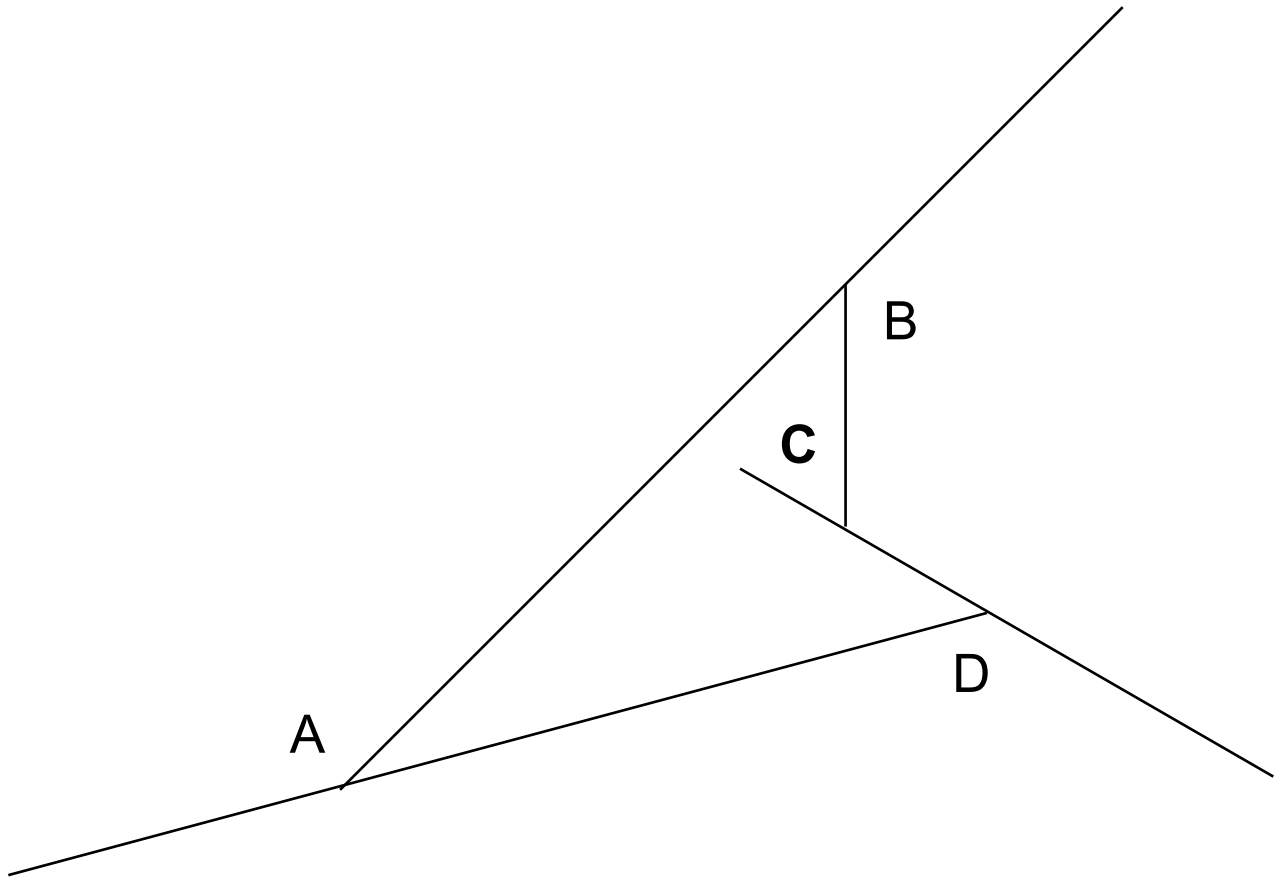
Quadrilateral 2



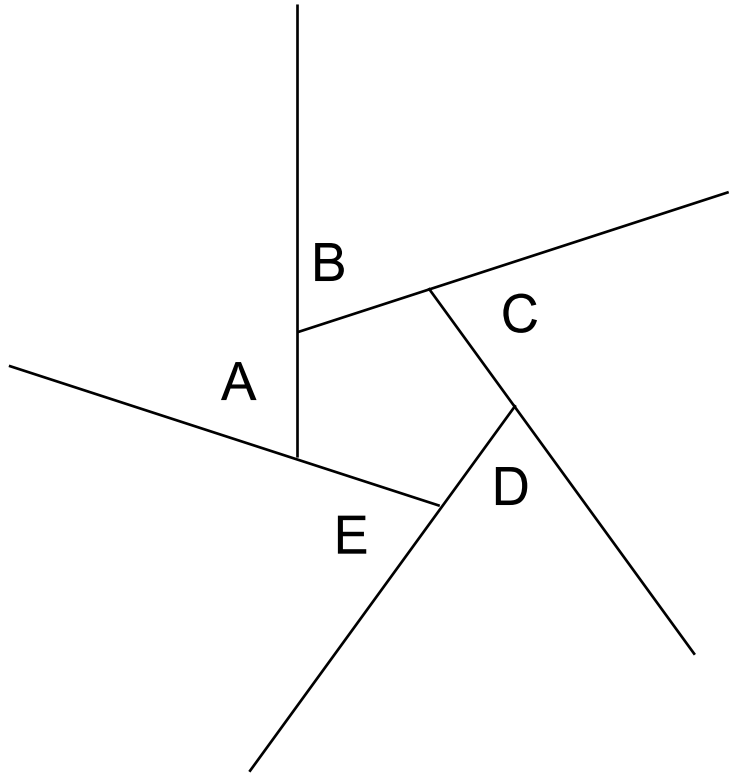
Quadrilateral 3



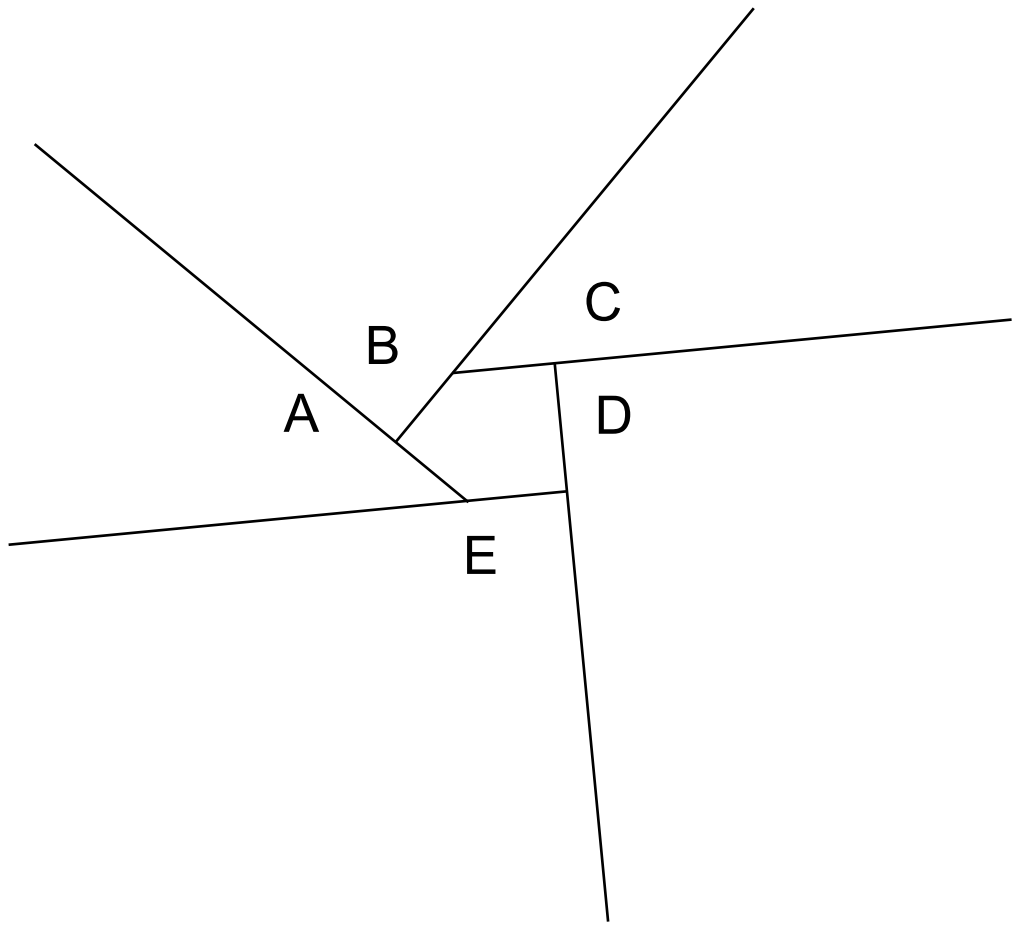
Quadrilateral 4
Notice that on a concave polygon, one of the “exterior” angles (C) lands inside the shape and its value should be treated as a **negative** number.



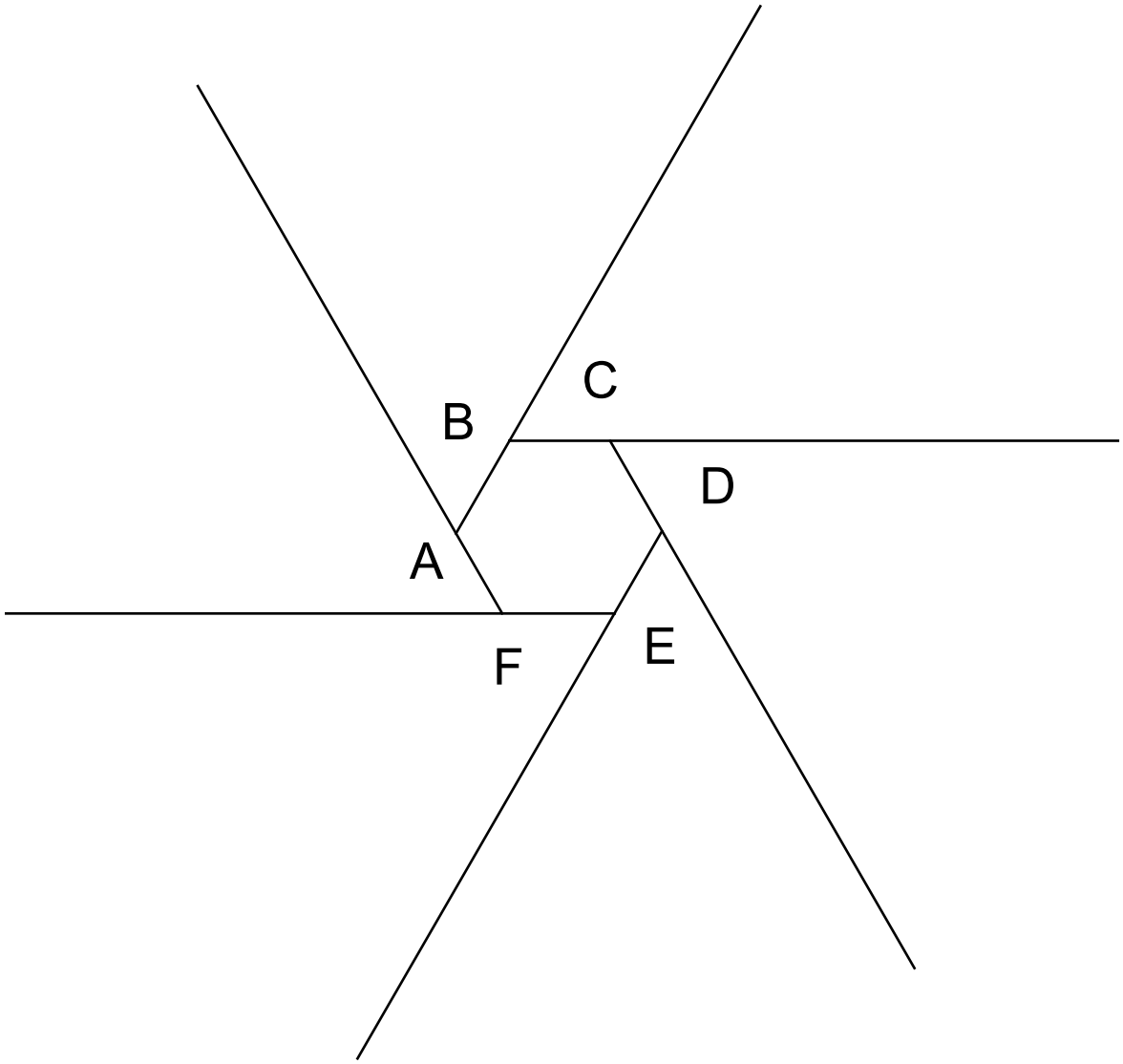
Pentagon 1



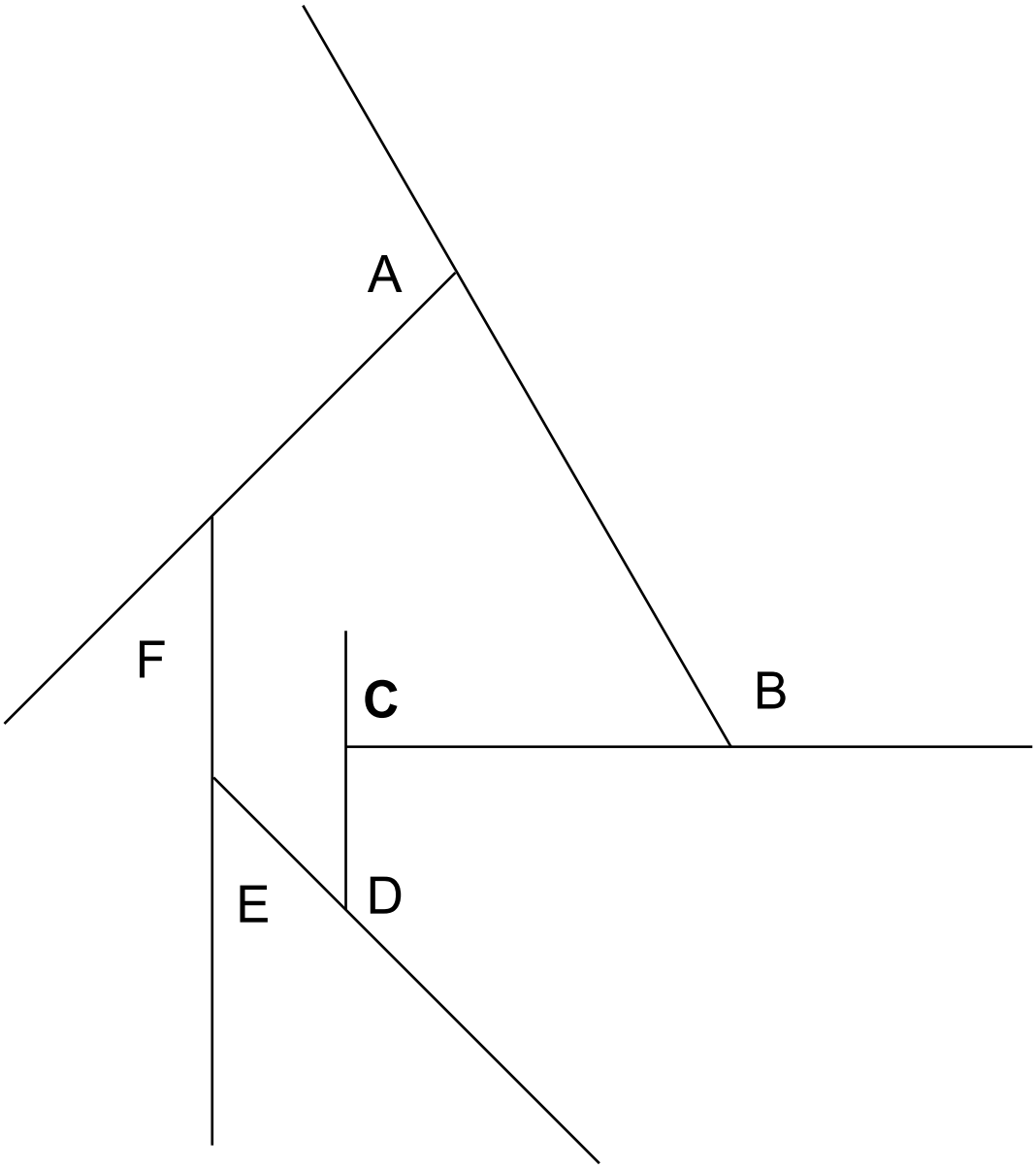
Pentagon 2



Hexagon 1



Hexagon 2
Notice that on a concave polygon, one of the “exterior” angles (C) lands inside the shape and its value should be treated as a **negative** number.



Level 2: Abstraction – Diagonals of Quadrilaterals

In this activity, students make observations (Analysis) about the diagonals of quadrilaterals. Then they test their observations across wider samples (Abstraction). You could also ask them to make observations about other properties of quadrilaterals such as parallel sides or opposite congruent angles.

For each type of quadrilateral, ask the students to construct the diagonals of the first example. Then ask them to make observations about the result.

Next have them test this on the subsequent examples.

Finally, you can have them test this on other types of quadrilaterals and make conjectures based on their observations.

Diagonals of Parallelograms

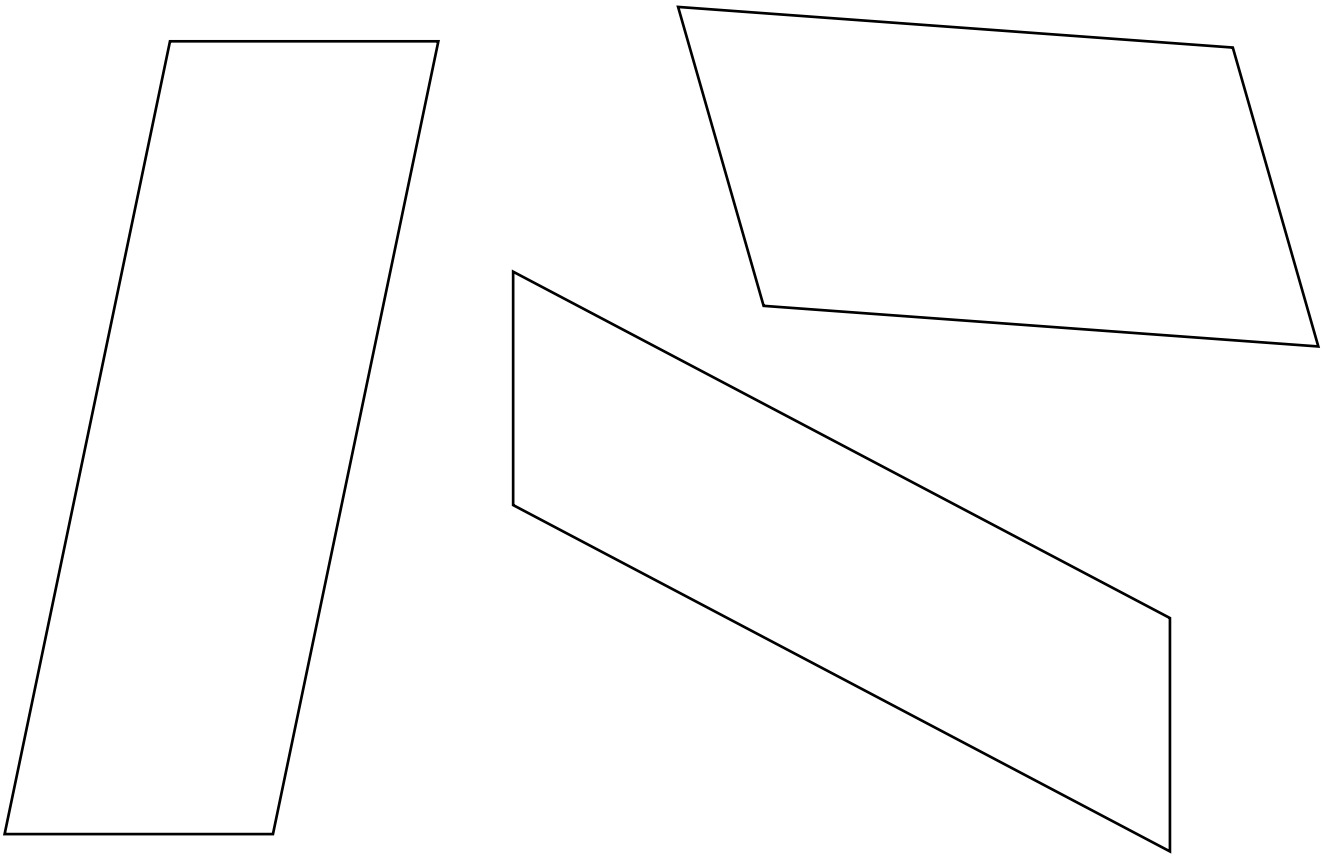
Name_____

Date_____ Class_____

Construct the diagonals of this parallelogram. Label their intersection point E. What do you notice about the lengths of AE and CE? What about the length of BE and DE?

_____

Now do the same for these other parallelograms. Is the result the same? _____



Diagonals of Rectangles

Name_____

Date_____ Class_____

Construct the diagonals of this rectangle. What do you notice about the lengths of the two diagonals? What do you notice about where they intersect?

_____

_____

Now do the same for these other rectangles. Is the result the same? _____

Diagonals of Squares

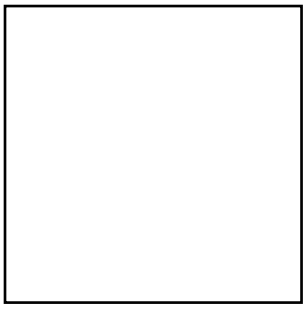
Name_____

Date_____ Class_____

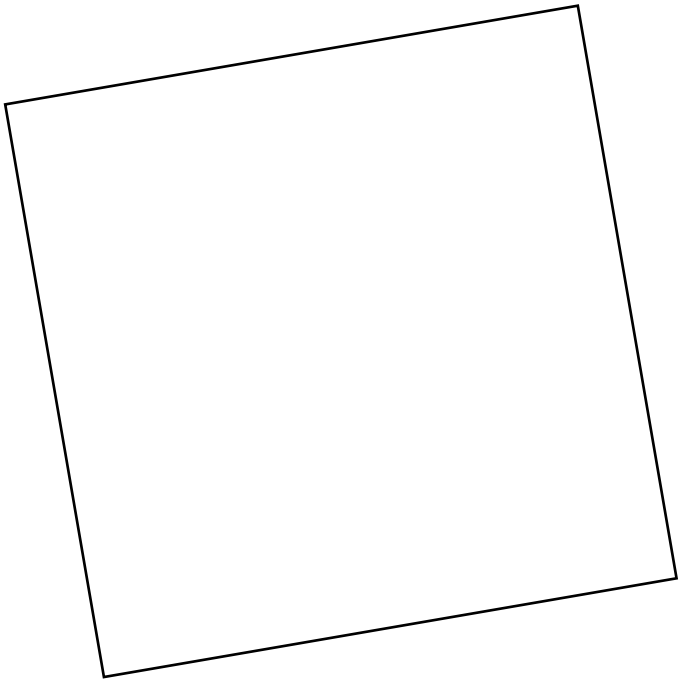
Construct the diagonals of this square. What properties do you notice about the diagonals?

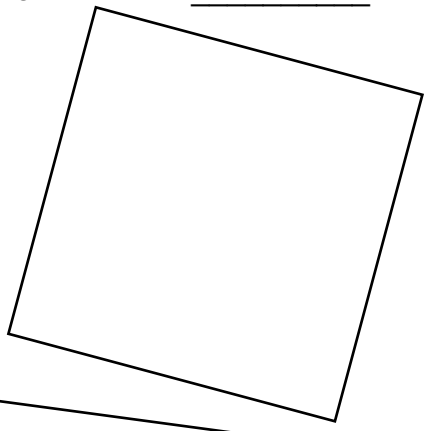
_____

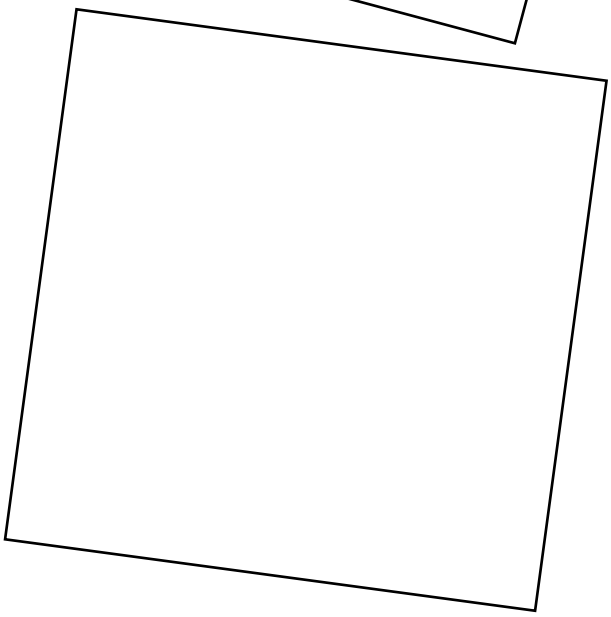
_____



Now do the same for these other squares. Is the result the same? _____







Diagonals of Rhombi

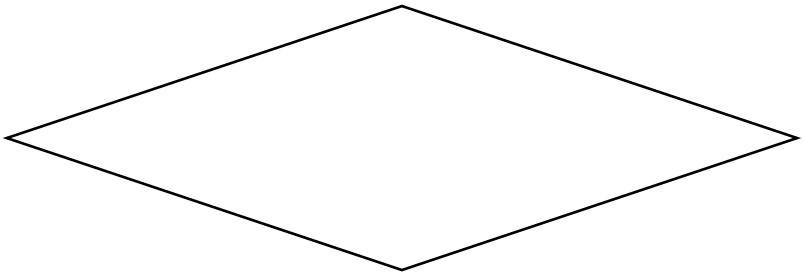
Name_____

Date_____ Class_____

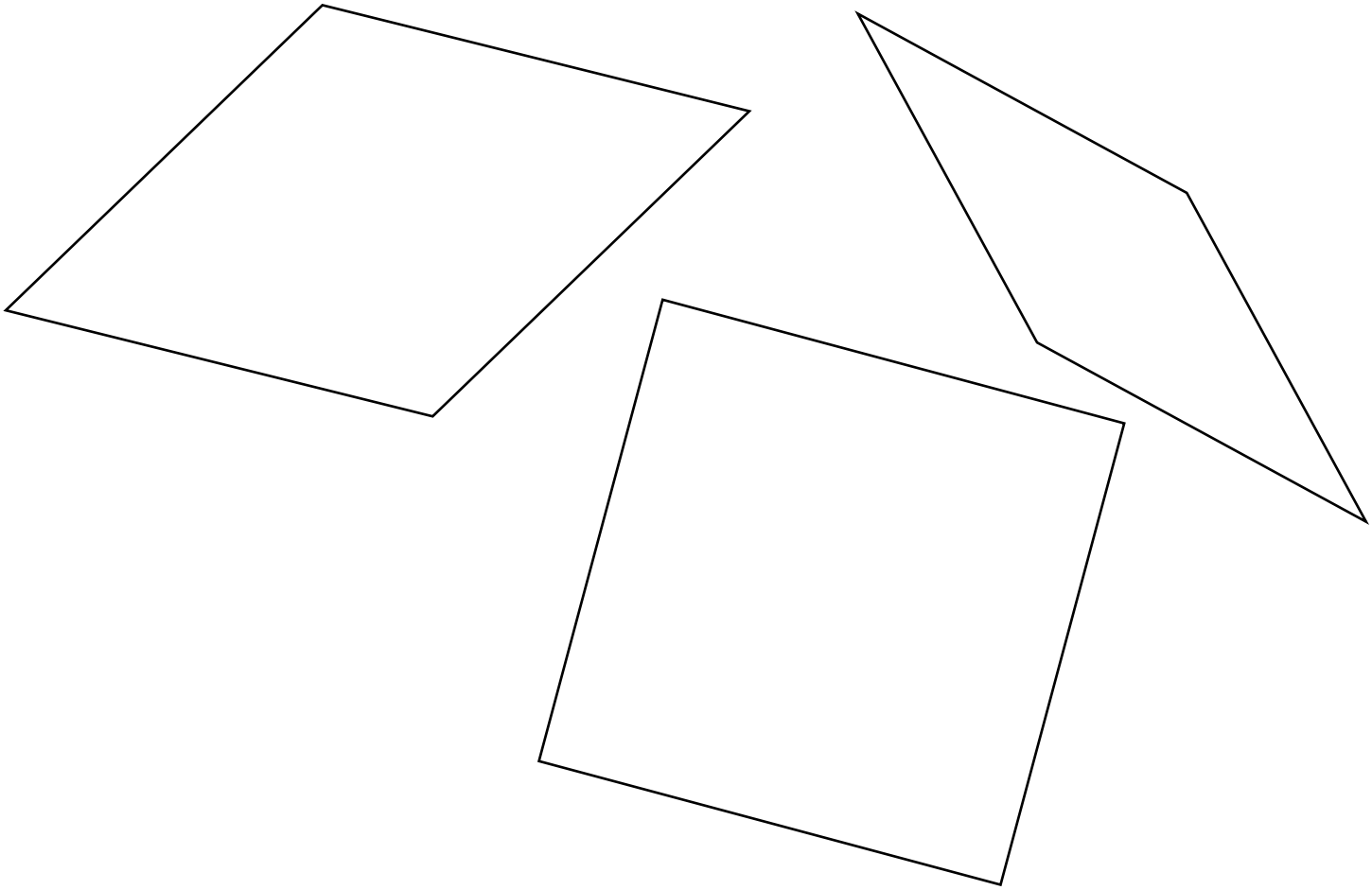
Construct the diagonals of this rhombus. What properties do you notice about the diagonals?

_____

_____



Now do the same for these other rhombi. Is the result the same? _____



Diagonals of Kites

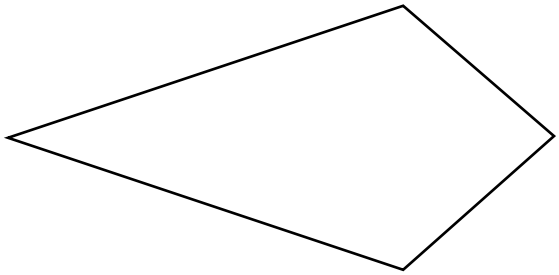
Name_____

Date_____ Class_____

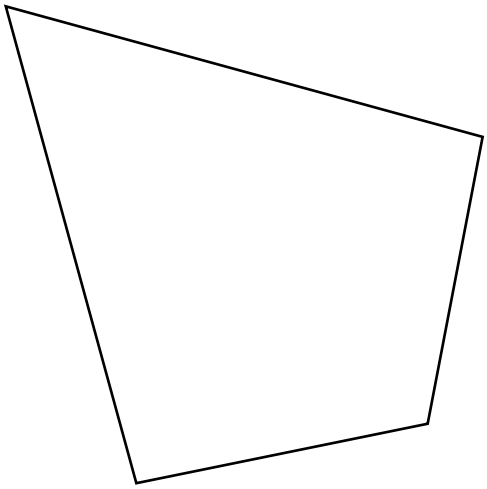
Construct the diagonals of this kite. What properties do you notice about the diagonals?

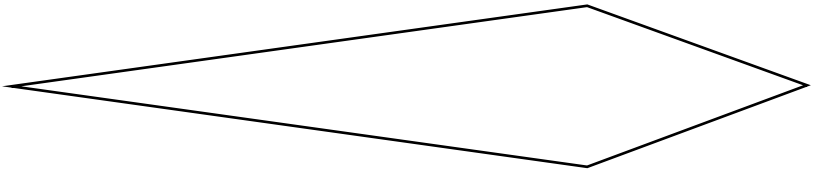
_____

_____

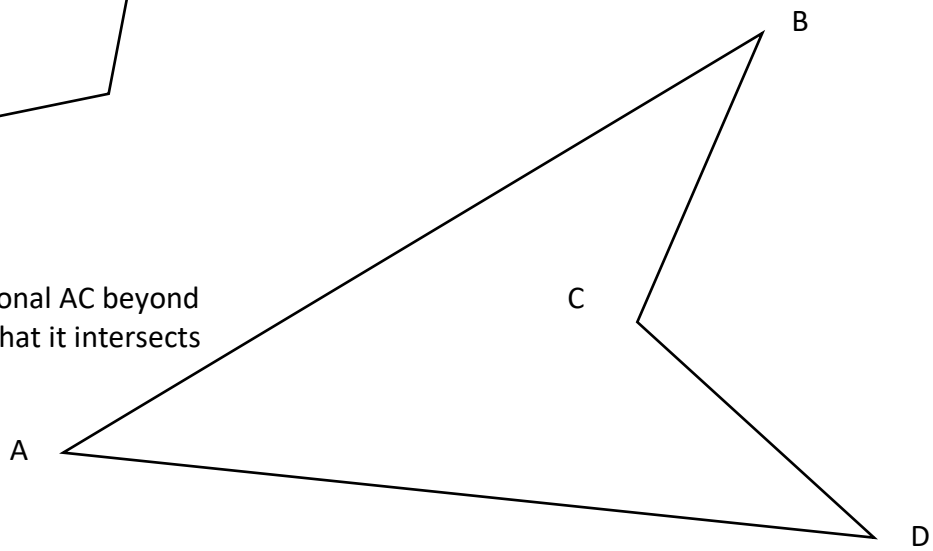


Now do the same for these other kites. Is the result the same? _____





Extend diagonal AC beyond the kite so that it intersects with BD.



Diagonals of Quadrilaterals

Name_____

Date_____ Class_____

1. A quadrilateral has diagonals that form perpendicular bisectors. What is the shape? Sketch and label it to show your solution.

_____

2. A quadrilateral has diagonals that bisect each other. What are some possible shapes? Sketch and label it to show your solution.

_____

3. A quadrilateral has congruent diagonals. What are some possible shapes? Sketch and label it to show your solution.

_____

4. A quadrilateral has one diagonal that is the perpendicular bisector of the other. What is the shape? Sketch and label it to show your solution.

_____

Diagonals of Quadrilaterals answer key

Diagonals of parallelograms

The diagonals bisect each other. The student may also notice that the vertical angles at the intersection are also congruent.

Diagonals of rectangles

The diagonals bisect each other since a rectangle is a parallelogram. Further, the diagonals are congruent.

Diagonals of squares

The diagonals are congruent and bisect each other since squares are rectangles and they are also parallelograms. The diagonals are also perpendicular bisectors of one another.

Diagonals of rhombi

The diagonals are perpendicular bisectors of one another, but they are not congruent unless the rhombus is a square.

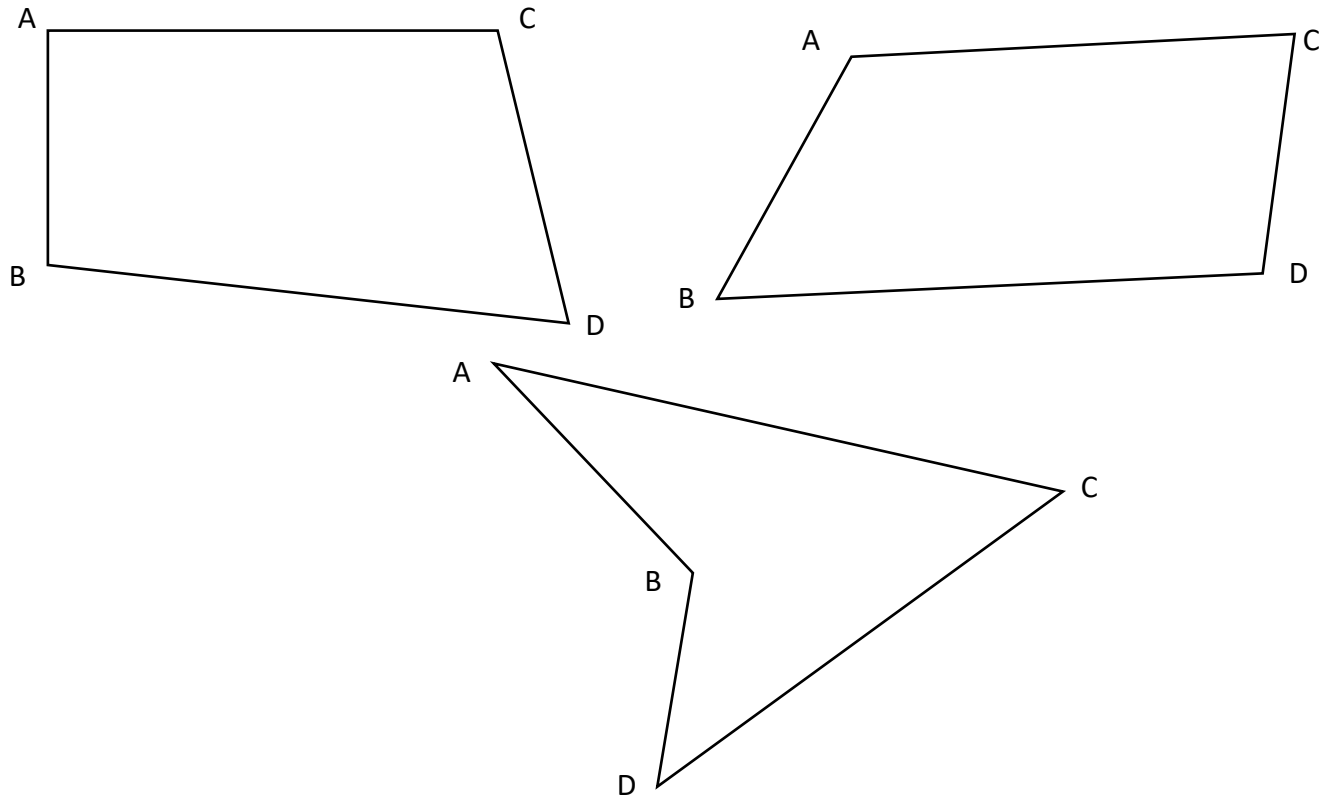
Diagonals of kites

One diagonal is the perpendicular bisector of the other.

Level 2: Abstraction – Midpoints of a quadrilateral

Again, we will give students an opportunity to explore the properties of a quadrilateral and extend their discovery across multiple examples. This activity lends itself to either formal geometric constructions or an easier method that is described below.

Give the students the activity page and ask them to place four points, A, B, C, and D at the locations of their choice. Here are some possible options.



Now ask the students to find the midpoints of each side. This can be done with formal compass and straightedge construction as mentioned above, but it can also be done very simply by folding vertex A to vertex B and creasing the paper.

Then have them join the midpoints to form a new quadrilateral. Have the describe their shape.

Ask them to compare their results to other students on their team. Then get a whole class response. Ask them to come up with a single statement that describes the results in general. The midpoints of any quadrilateral will form a parallelogram. In some cases, the parallelogram might be a rectangle, rhombus, or square, but these are all parallelograms.

Midpoints of A Quadrilateral

Name_____

Date_____ Class_____

Mark four points in any location on your paper. They should be space out – not too close together. Label the points A, B, C, D. Connect them to form a quadrilateral.

Find the midpoints of each side of your quadrilateral.

Connect the midpoints to form a new quadrilateral. What shape is the quadrilateral formed by these midpoints?

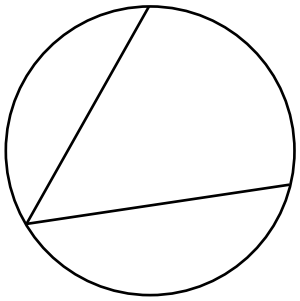
_____

Try this again on the back of the page. Did you get the same result? _____

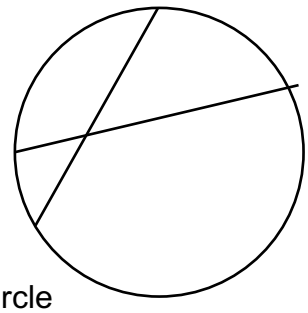
Level 2: Abstraction – Angles within Circles

We'll use this same approach of making observations about properties and characteristics (Analysis) and extending that through testing (Abstraction) as we investigate the angles formed by chords within circles. We will explore three different scenarios.

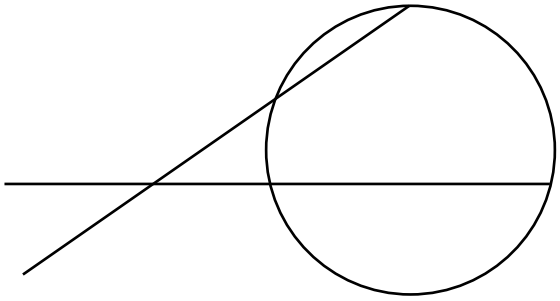
Inscribed angles



Chords intersecting within a circle



Chords intersecting outside a circle



The circle used in this activity is divided into 36 arcs. Thus, each arc represents 10º. This helps prevent mismeasurement. All measurements will be in increments of 10º or 5º. Later students can explore circles in which angles can take any measure.

Angles within Circles

Name_____

Inscribed Angles

Date_____ Class_____

Since a circle has 360º, how many degrees does each arc represent? _____

Construct angle ABC. How many degrees are in arc AC? _____

How many degrees are in angle ABC? _____

Construct angle BAC. How many degrees are in arc BC? _____

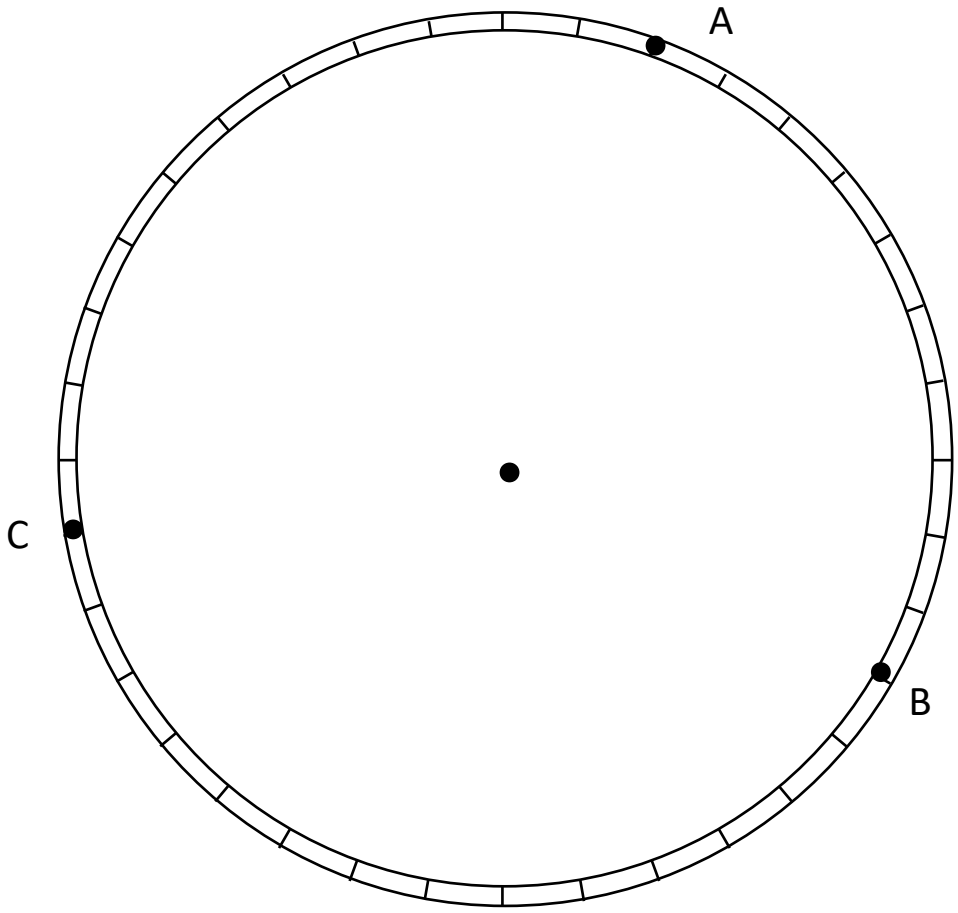
How many degrees are in angle BAC? _____

Now, construct angle ACB. How many degrees are in arc AB? _____

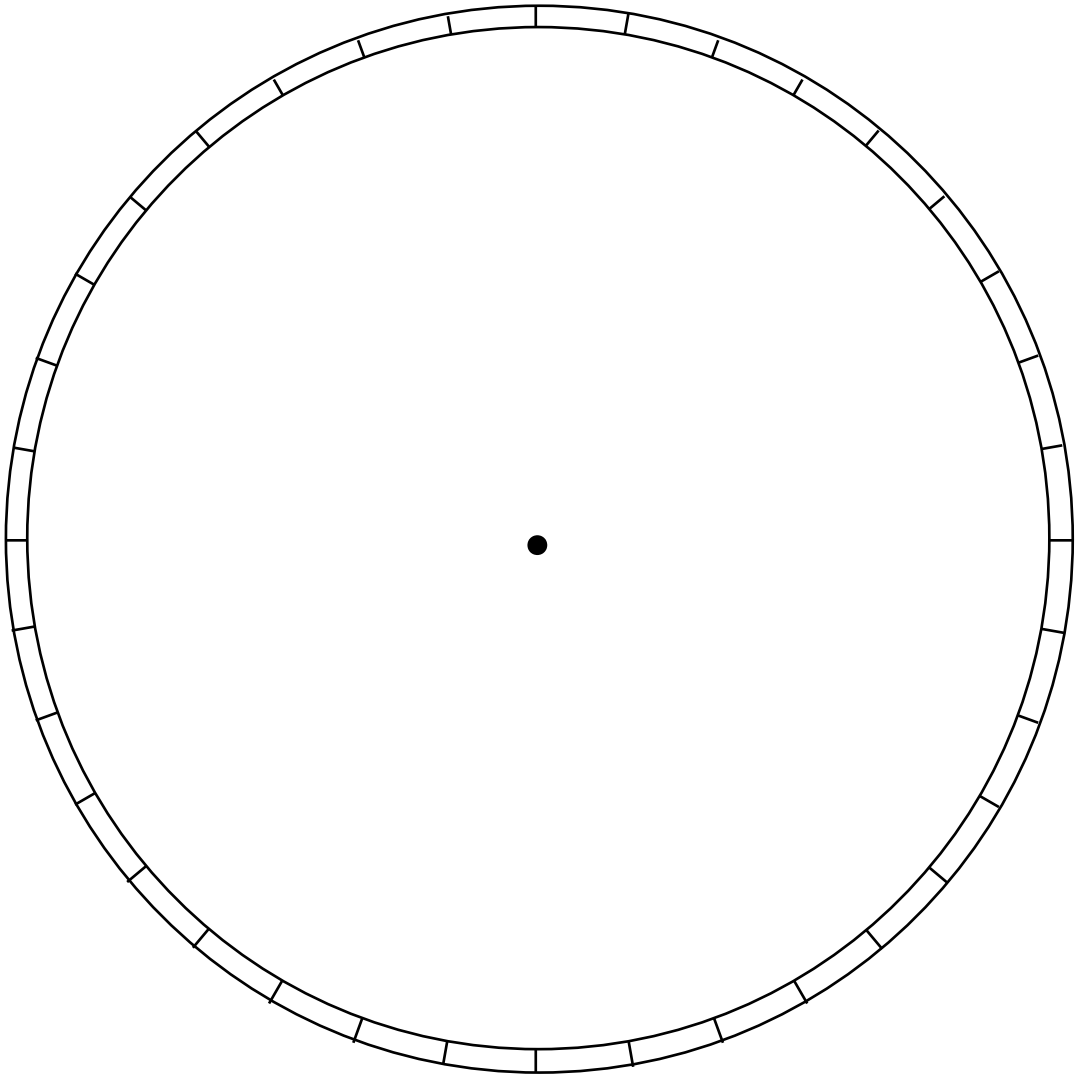
How many degrees are in angle ACB? _____

Summarize your observation: _____

_____



Now try this again by placing three points in new locations on the circle.



Does the relationship still work? _____

Did it work for other students in your group? _____

Write a sentence summarizing the relationship between the measure of the arc and the inscribed angle.

_____

_____

Angles within Circles

Name_____

Internal Angles

Date_____ Class_____

Now the chords intersect inside the circle.

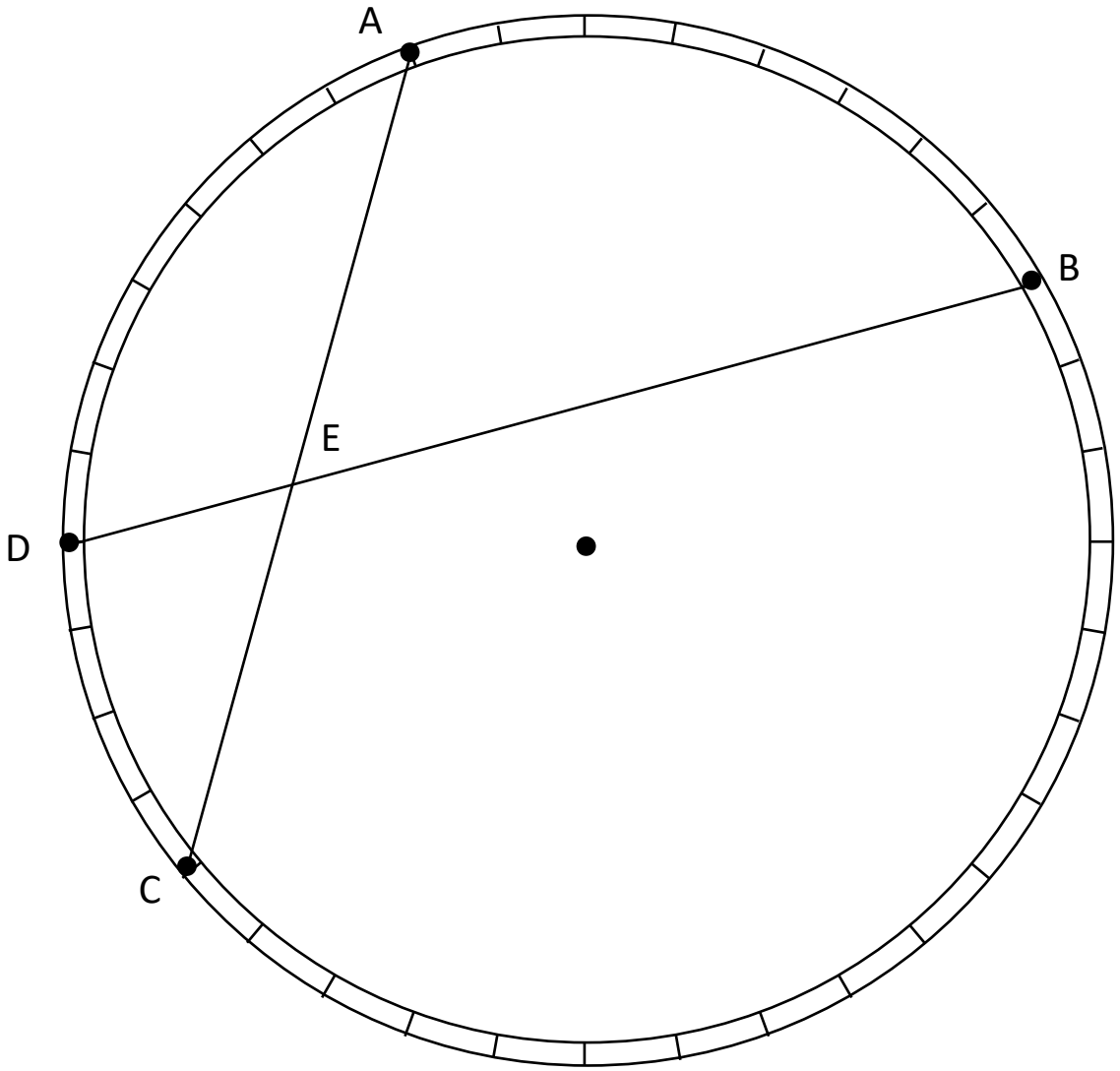
How many degrees are in arc AB? _____

How many degrees are in arc CD? _____

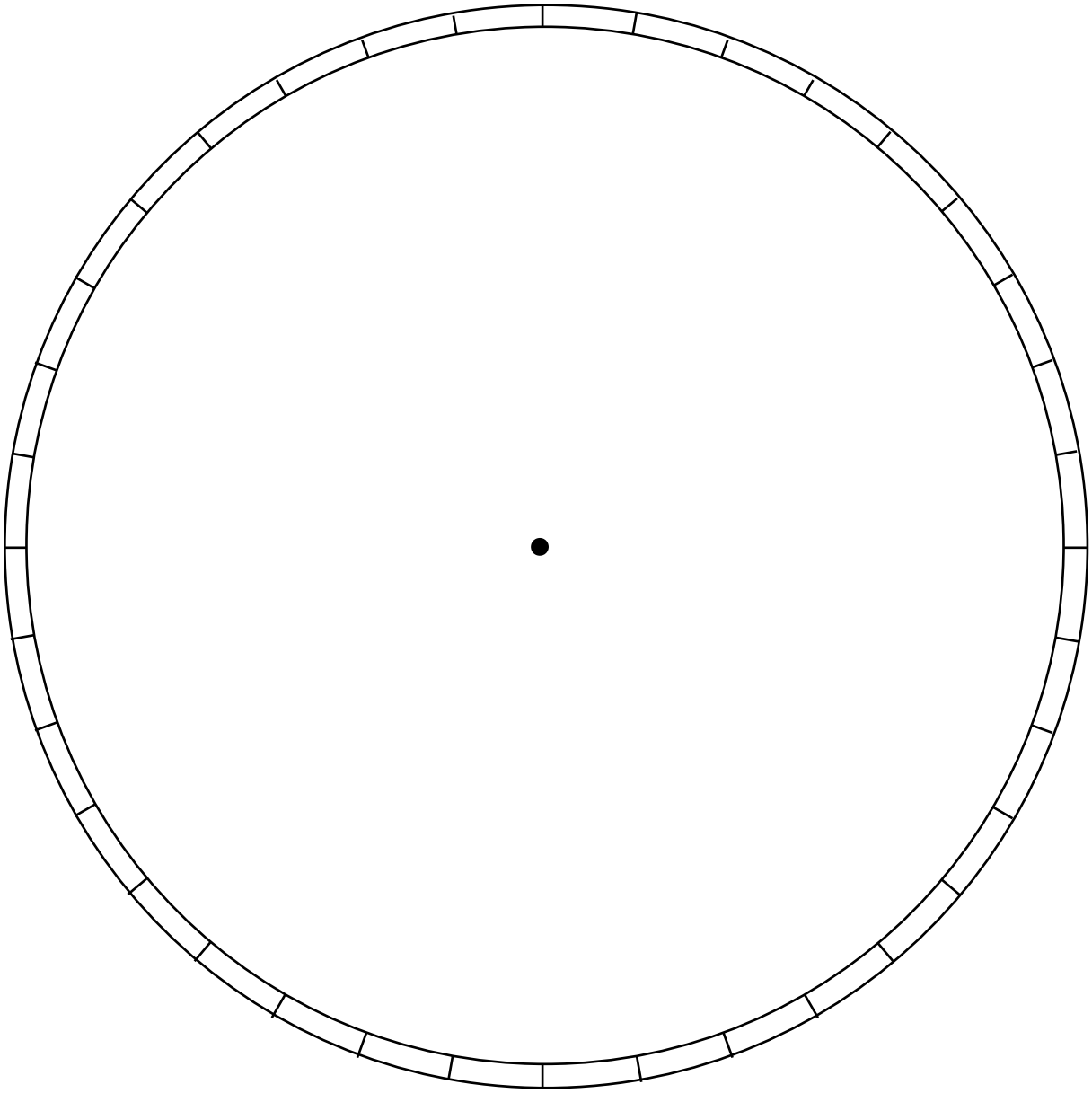
What is the measure of angle AEB? _____

How does the measure of angle AEB compare to the two arcs?

_____



Now try this again by picking four points of your own.



Does the relationship still hold? _____

Did it work for the other students in your group? _____

Write a formula to summarize this relationship. _____

Angles within Circles

Name_____

External Angles

Date_____ Class_____

Now the chords intersect outside the circle.

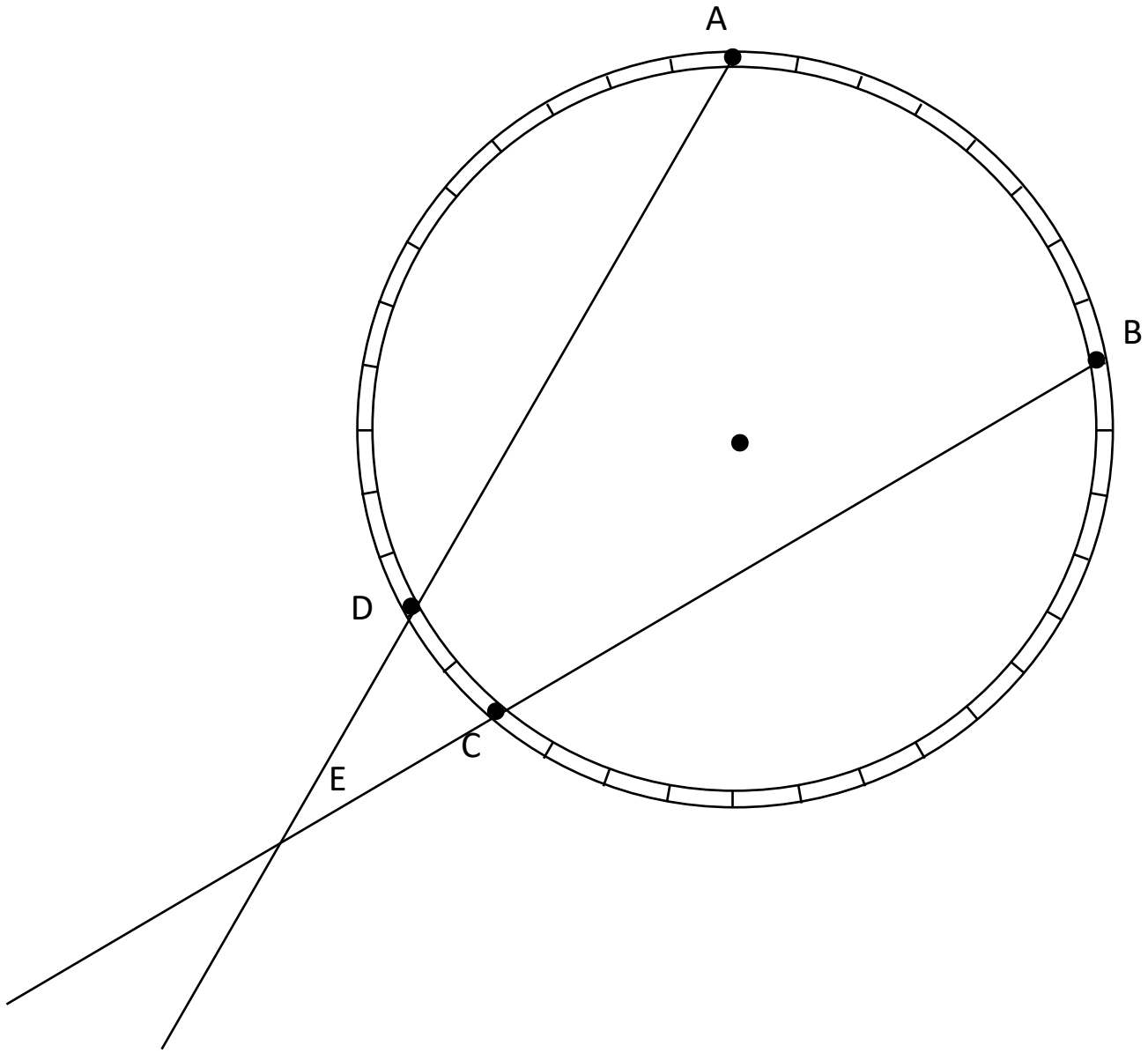
How many degrees are in arc AB? _____

How many degrees are in arc CD? _____

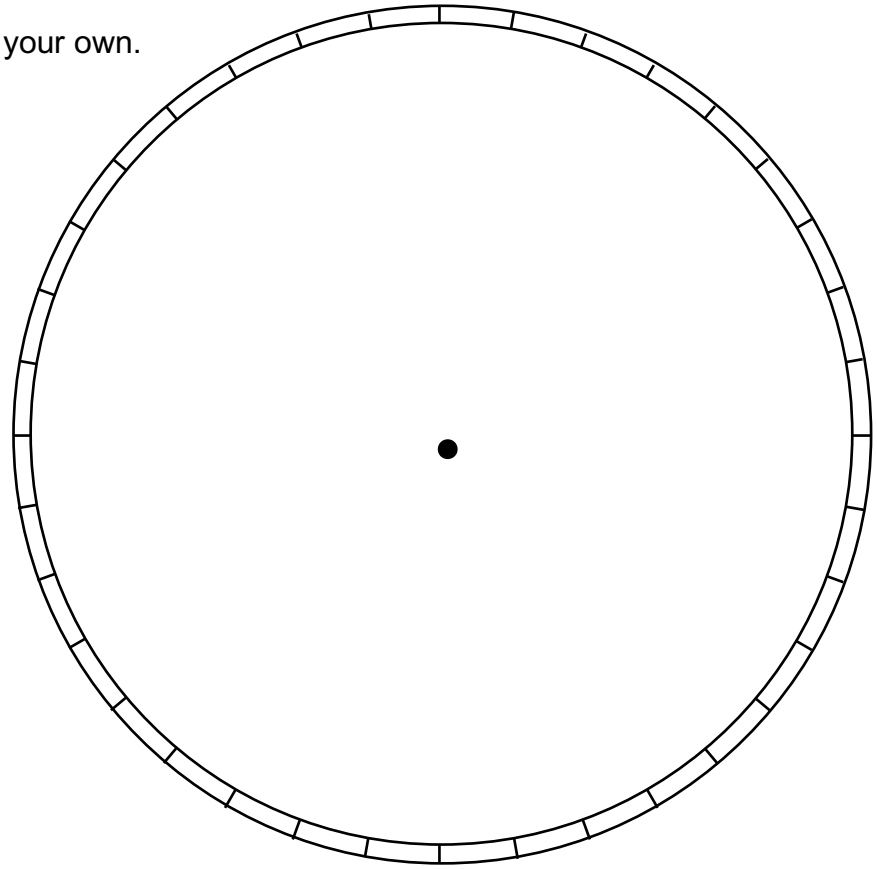
What is the measure of angle AEB? _____

How does the measure of angle AEB compare to the measures of the two arcs?

_____



Now repeat this with four points of your own.



Does the relationship still hold? _____

Did it work for the other students in your group? _____

Write a formula to summarize this relationship. _____

Angles within Circles answer key

Inscribed angles

There are 36 tick marks, so each tick mark represents 10º

Arc AC measures 120º.

Angle ABC measures 60º.

Arc BC measures 140º.

Angle BAC measures 70º.

Arc AB measures 100º

Angle ACB measures 50º

Yes, it always works.

The inscribed angle is half the arc that it inscribes.

Internal angles

Arc AB measures 80º.

Arc CD measures 40º.

Angle AEB measures 60º.

Angle AEB is half the sum of the two arcs. It is the average or mean.

Yes, the relationship always works.

Angle AEB = (Arc AB + Arc CD)/2

External angles

Arc AB measures 80º.

Arc CD measures 20º.

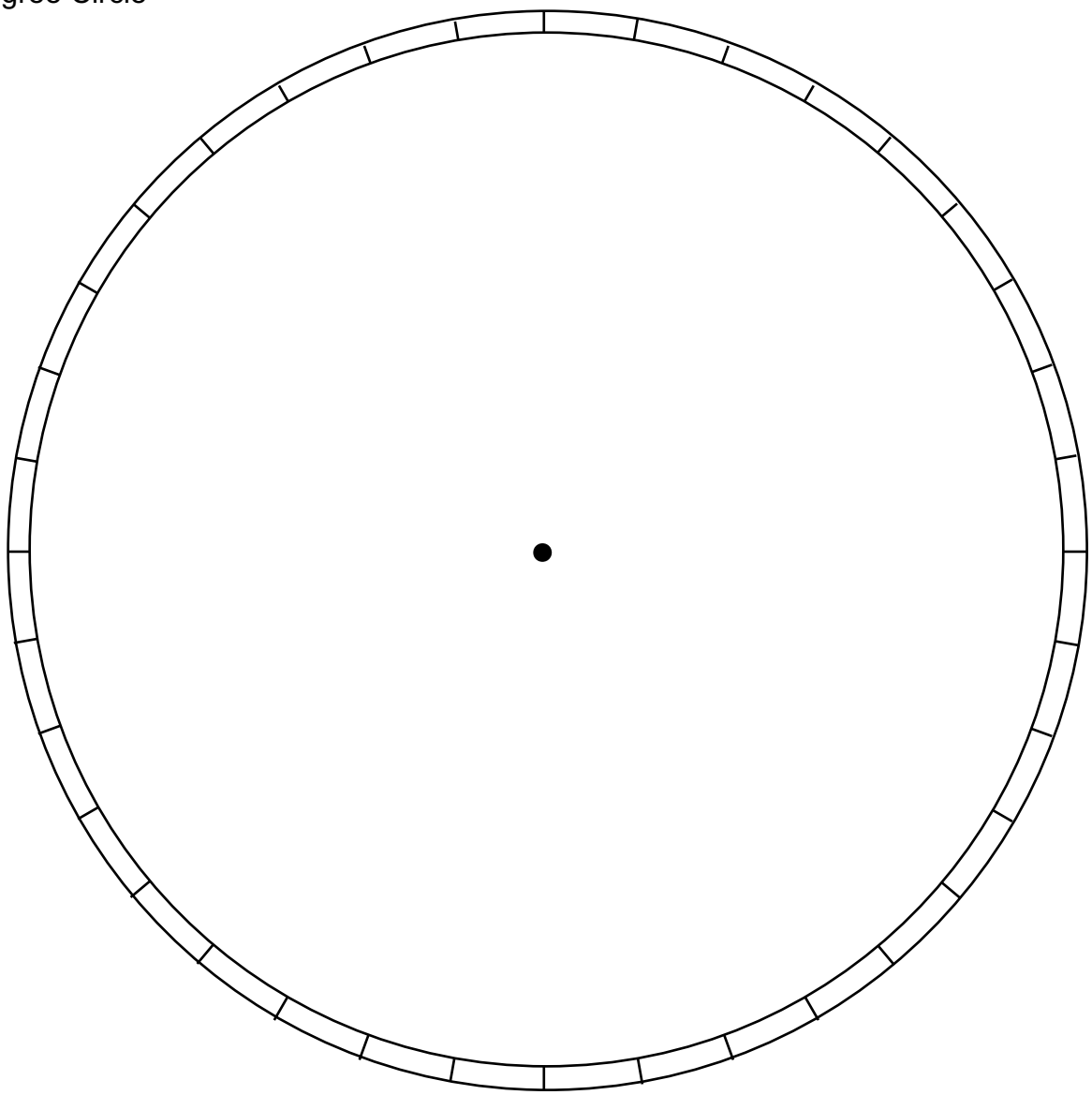
Angle AEB measures 30º.

Angle AEB is half the *difference* between the two arcs.

Yes, the relationship always works.

Angle AEB = (Arc AB – Arc CD)/2

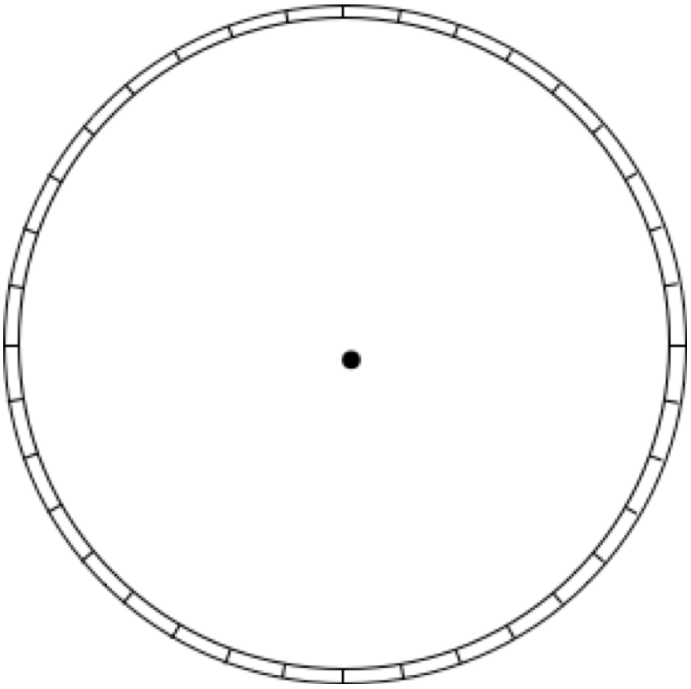
Degree Circle

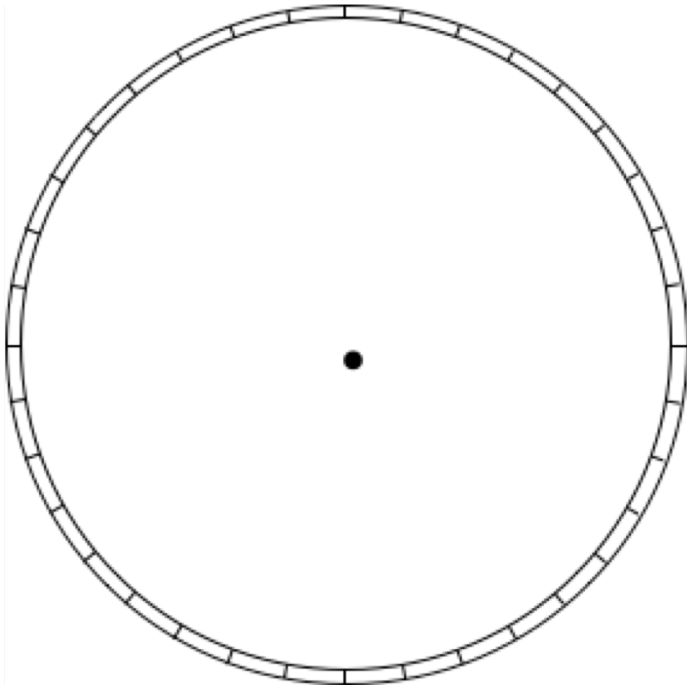


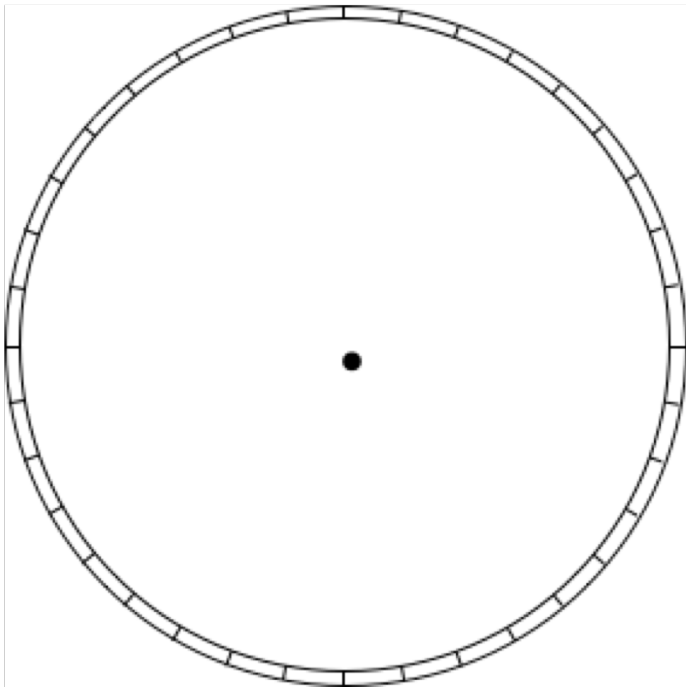
Degree Circles

Name_____

Date_____ Class_____







Transitioning from Level 2 to Level 3: Deduction

In level 2, the student uses *inductive reasoning* to draw conclusions and form conjectures about the properties of geometric shapes. This has been based on numerous but limited examples. The characteristics and observations the student have seen seem to hold without exception, but they have only tried a few examples.

The fact that the other students in class had the same results and came to the same conclusion lends to the assurance that these observations are absolute. But we are far from proving that.

Again, I remind my students that geometry is in some ways like a trial. We want to be *absolutely* sure. And just as we can come up with an infinite number of examples to show that prime numbers are odd, there exists an exception to prove us wrong.

There are two ways to move beyond the limited number of examples that the student has explored. One is to use deductive proof, but there is a smaller and more incremental step that we can take in our journey to ensure that more students reach the final step in our ascension.

That is to explore a *multitude* of examples. Online software such as GeoGebra allows a student to test their investigation across a wider sampling. We will look at some lessons in GeoGebra that extend the learning we have encountered in this manual.

Interior and Exterior Triangle Angle Sums

Name_____

Date_____ Class_____

Go to www.geogebra.org/m/FAhtKpR5

Move the vertices of the triangle. Then move the slider.

What is the interior angle sum? _____

Make a new type of triangle by moving the vertices. Move the slider.

What is the interior angle sum now? _____

Can you make a triangle that does not do this? _____

Now select “Exterior Angles”.

Create a new triangle. Then move the slider.

What is the exterior angle sum? _____

Make another triangle. Move the slider.

What is the result? _____

Do one more triangle, and move the slider.

What is the exterior angle sum?

_____

Interior and Exterior Triangle Angle Sums answer key

Move the vertices of the triangle. Then move the slider.

What is the interior angle sum? 180º

Make a new type of triangle by moving the vertices. Move the slider.

What is the interior angle sum now? 180º

Can you make a triangle that does not do this? No

Now select “Exterior Angles”.

Create a new triangle. Then move the slider.

What is the exterior angle sum? 360º

Make another triangle. Move the slider.

What is the result? 360º

Do one more triangle, and move the slider.

What is the exterior angle sum?

360º

Quadrilaterals And Midpoints

Name_____

Date_____ Class_____

Go to www.geogebra.org/m/VkxdAZrG

Move the vertices to form a square. What shape is formed by the midpoints?

_____

Now move the vertices to form a rectangle. What shape is formed by the midpoints now?

_____

Create a rhombus. What shape to the midpoints make?

_____

Make a parallelogram. What shape to the midpoints make?

_____

What if you make an isosceles trapezoid? Sketch it here. What shape to the midpoints make?

_____

Now make a right trapezoid. Sketch it below. What shape to the midpoints make?

_____

Create and irregular quadrilateral with no congruent or parallel sides. Sketch it. What shape to the midpoints make?

_____

Make a concave quadrilateral. Sketch it. What shape to the midpoints make?

_____

Quadrilaterals and Midpoints answer key

Move the vertices to form a square. What shape is formed by the midpoints?

Square

Now move the vertices to form a rectangle. What shape is formed by the midpoints now?

Rhombus

Create a rhombus. What shape to the midpoints make?

Rectangle

Make a parallelogram. What shape to the midpoints make?

Parallelogram

What if you make an isosceles trapezoid? Sketch it here. What shape to the midpoints make?

Rhombus

Now make a right trapezoid. Sketch it below. What shape to the midpoints make?

Parallelogram

Create and irregular quadrilateral with no congruent or parallel sides. Sketch it. What shape to the midpoints make?

Parallelogram

Make a concave quadrilateral. Sketch it. What shape to the midpoints make?

Parallelogram

Diagonals of Parallelograms

Name_____

Date_____ Class_____

Go to www.geogebra.org/m/GFwZ5qdf#material/YT2AVyyp

Select “Show diagonals”. Then select “Show additional angle measures.”

Move the vertices to form a parallelogram of your design. What do you notice about the lengths of the diagonals?

_____

Create a new parallelogram. Does your observation still hold?

_____

Make a rectangle. Does the relationship still work?

_____

What if you make a rhombus?

_____

What happens if you form a square?

_____

There are eight questions below the animation. Write your answers here:

1. _____

2. _____

3. _____

4. _____

5. _____

6. _____

7. _____

8. _____

Diagonals of Parallelograms answer key

Select “Show diagonals”. Then select “Show additional angle measures.”

Move the vertices to form a parallelogram of your design. What do you notice about the lengths of the diagonals?

They bisect each other.

Create a new parallelogram. Does your observation still hold?

Yes

Make a rectangle. Does the relationship still work?

Yes

What if you make a rhombus?

It still works.

What happens if you form a square?

The diagonals still bisect each other.

There are questions below the animation. Write your answers here:

1. Yes

2. Yes

3. Yes

4. Not always

5. Not always

6. Not always

7. Yes, both do

Teaching Area Formulas
Making Geometry Make Sense!

Overview:

Students will move through a seamless transition from finding the areas of squares to rectangles, parallelograms, triangles, and trapezoids. The area formula for each polygon will build upon those they have already learned ensuring more retention and understanding of the formulas.

Then they will apply these formulas to find the areas of complex shapes — the areas of states — as they apply their skills to problem-solving situations.

Required Materials:

- ☒ Geoboard paper
- ☒ Activity master
- ☒ Centimeter rulers

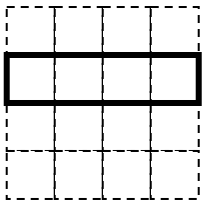
Optional Materials:

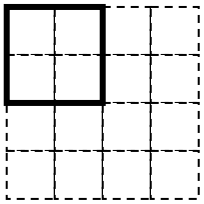
- ☐ Geoboards
- ☐ Calculators

Procedure:

1 A lot of material is covered in this activity. You may wish to spend more than one day exploring it. The first day might be spent deriving area formulas for the rectangle, square, and parallelogram since they can be written as identical formulas. A second day can be spent on the triangle and a third day on the trapezoid. Your knowledge of your students is your best guide.

2 Pass out geoboard paper and, if you wish, geoboards and rubber bands to each student. Ask them to try to find all the rectangles that have an area of four square units. They should build these on their geoboards if they have them and sketch them on their paper. There are only two solutions: a rectangle with a base of four units and a height of one unit and a square with a side length of two units as shown. Build these on your geoboard display master. Some students may say that a rectangle with a base of one unit and a height of four units is another solution, and you may wish to include this. Generally, you do not want to include slides, flips and rotations as separate solutions in this activity.



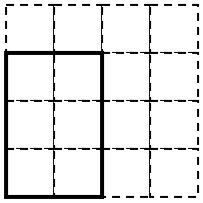


Rectangles with areas of four square units.

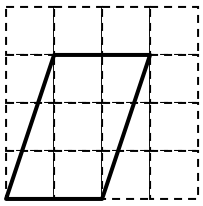
3 Next ask them to build and sketch all the rectangles with an area of six square units. Now only a two by three (and a three by two) can be built on the geoboard since there is not enough room to build one with a base or height of six units. Build the two by three rectangle on your display geoboard. Write the following questions on the board:

- We can see that the 2x3 rectangle has six square units inside. How could we arrive at that answer without counting them? What if the rectangle were a 4x3 or a 10x5?

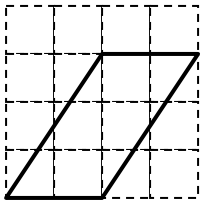
Some students will see that the area of a rectangle can be derived by multiplying its base and height. It is important to write the above questions on the board so the students see that a two by three rectangle can be written as a 2x3 rectangle. This will help them to see the formula.



4 Ask the class if this formula also works on the 2x2 square? They will see that it does. You may wish for the sake of simplification to say that the area formula for a square is also base times height. With older students, you should explain that since both sides of a square are equal, the area formula is generally stated as side times side, or s^2.

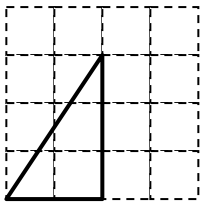


5 Tell the class that you will now transform the 2x3 rectangle into a parallelogram as shown in the margin. Ask them to find the area. They can see that the area is still six square units. Ask them to build and sketch other parallelograms that have an area of six square units. A second example is shown in the margin. Ask them what has remained the same besides the area. They can see that the base is still two. The height is also still three. What has changed is the length of the sides, but this has no effect on the area. To determine the area of a parallelogram, they only need to multiply the base and height, just like a square or rectangle.



$$A = b \bullet h$$

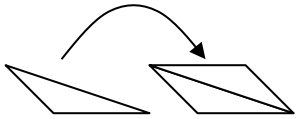
6 Now transform the last parallelogram into the triangle shown and ask them to find the area. Some will count regions and see that the area is three square units. Others may notice that it is half the previous parallelogram. It is important that the students are aware of both of these observations. Ask them to build and sketch all the triangles that have an area of three square units and a base of either two or three units. This will lead them



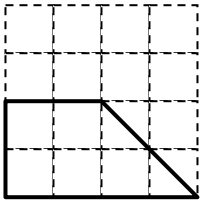
to find triangles that are halves of the parallelograms they just sketched. They will then better understand why the area formula for a triangle is half the area formula for a parallelogram.

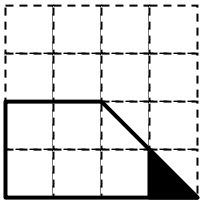
$$A = \frac{b \bullet h}{2}$$

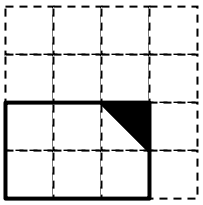
You may wish to show that any triangle can be duplicated and rotated to form a parallelogram that has twice the area.



7 Now form the trapezoid shown in the margin. The students should build and sketch it and find its area. They will see that it has an area of six square units also. However, its lower base is four and its height is two. Since 4 x 2 ≠ 6, the parallelogram formula will not work. If the geoboard is turned upside down, the base is two and the height is two, but since 2 x 2 ≠ 6, the formula still does not work. Show the students that the bottom right corner can be removed and used to fill the top right corner. This forms a rectangle with the same area. It has a base of three and a height of two, and now the area formula works. Thus you have to find the *average of the two bases* of a trapezoid and multiply that by the height.

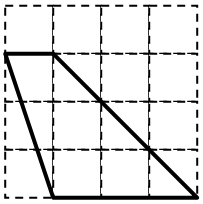






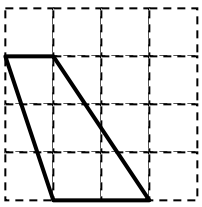
$$A = \frac{b_1 + b_2}{2} \bullet h$$

Ask the students to build and sketch all trapezoids that have an area of six square units. One of the most unusual solutions is shown in the margin. You may wish to ask students to explain why this is a trapezoid and how they can prove it has an area of six square units.



Journal Prompts:

Is it possible to design a trapezoid on the geoboard that has an area that is not a whole number of square units, such as 4.5? Why or why not? (Yes. See the margin.)



Can a square with an area of six square units be built on the geoboard? Why or why not? (No, it can't, because six is not a square number.)

Homework:

Give the students graph paper or dot paper. Ask them to find all the rectangles, parallelograms, triangles, and trapezoids that have an area of ten square units. This allows them to use a base or height that is greater than four units.

The assignment described below in "Taking a Closer Look" can also be used as a more challenging homework assignment.

Taking a Closer Look:

Have students use graph paper to make stylized maps of states. Subdivide the states into triangles, rectangles, squares, parallelograms, and trapezoids and find the areas. What would each square unit need to be to approximate the actual area of the state? Some sample states are included in the following pages and can be used as homework or classwork practice also. The six states increase in difficulty.

Assessment:

Allowing students to work in small teams on this task helps them check their results with one another. After each exploration, invite students to show their solutions on the board.

Answer key for state areas:

State	Area in square cm.	Area in square mi.
Nevada	166.5	109895
Minnesota	154.5	79617
New York	154	47224
Texas	80	261914
North Carolina	87	48718
California	157.5	155973

The Common Core Connection:

Third Grade:
CCSS.MATH.CONTENT.3.MD.C.7.B
Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

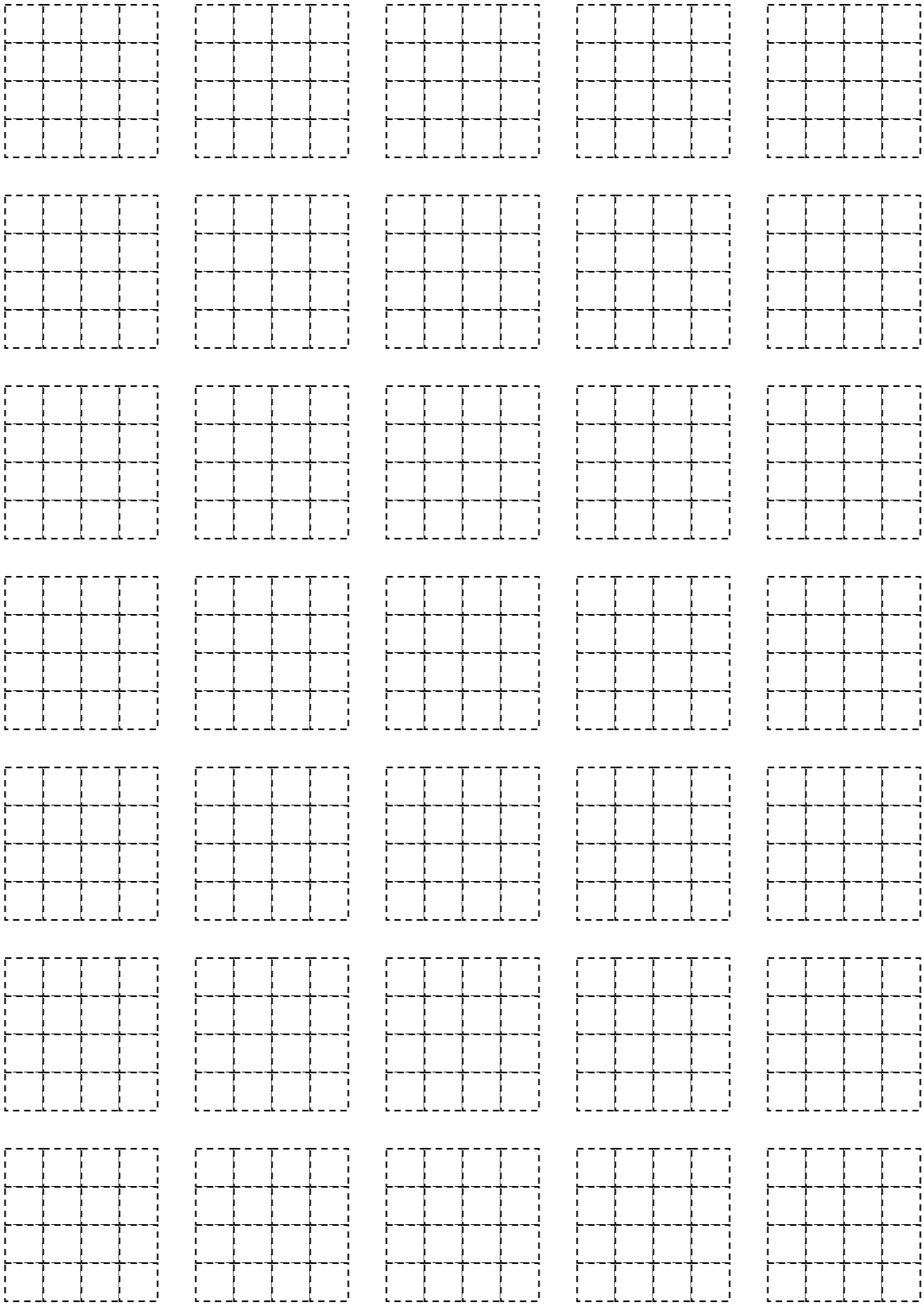
Fourth grade:
CCSS.MATH.CONTENT.4.MD.A.3
Apply the area and perimeter formulas for rectangles in real world and mathematical problems. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor*.

Fifth grade:
CCSS.MATH.CONTENT.5.G.B.3
Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.

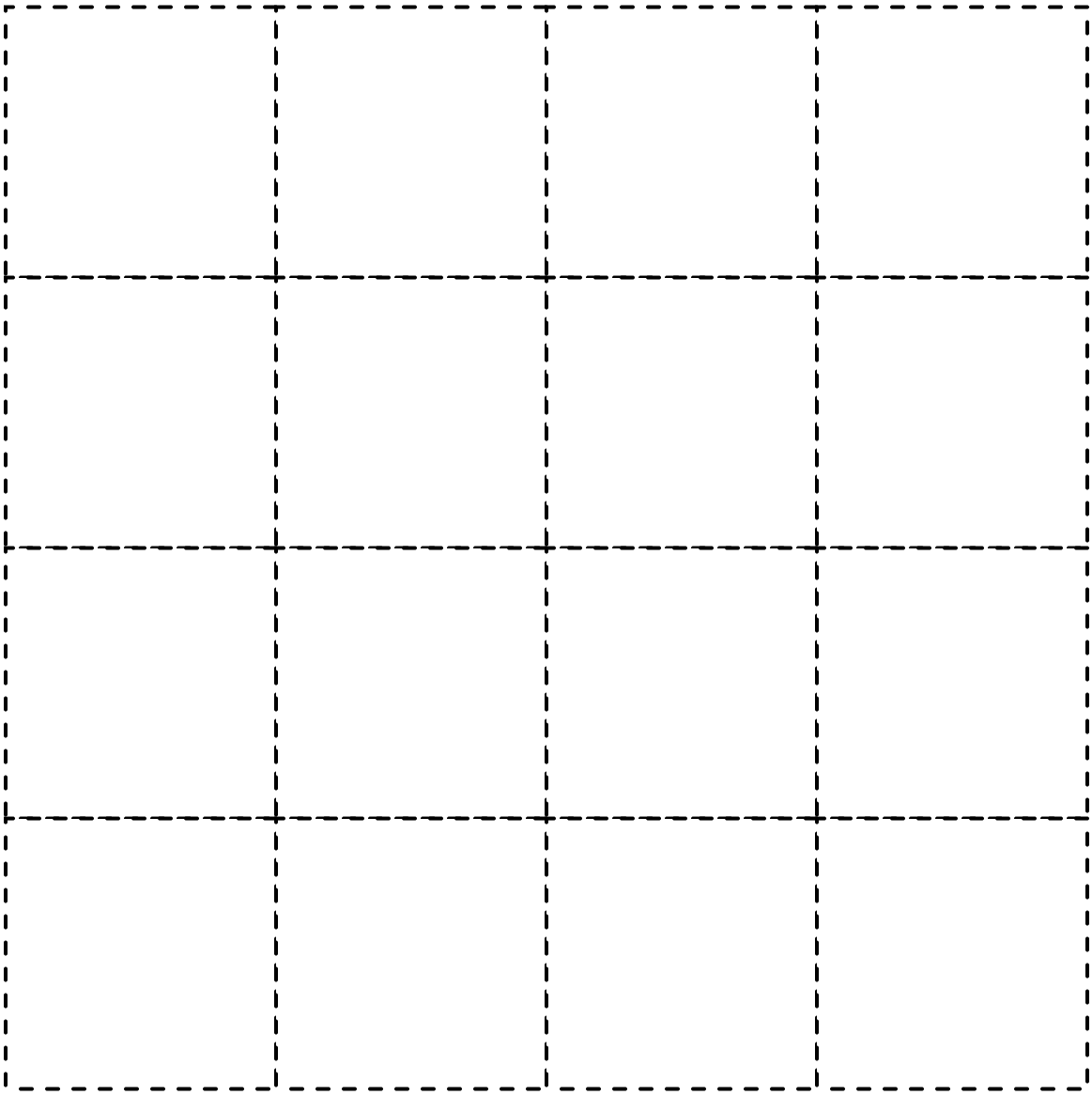
Sixth grade:
CCSS.MATH.CONTENT.6.G.A.1
Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Seventh grade:
CCSS.MATH.CONTENT.7.G.B.6
Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

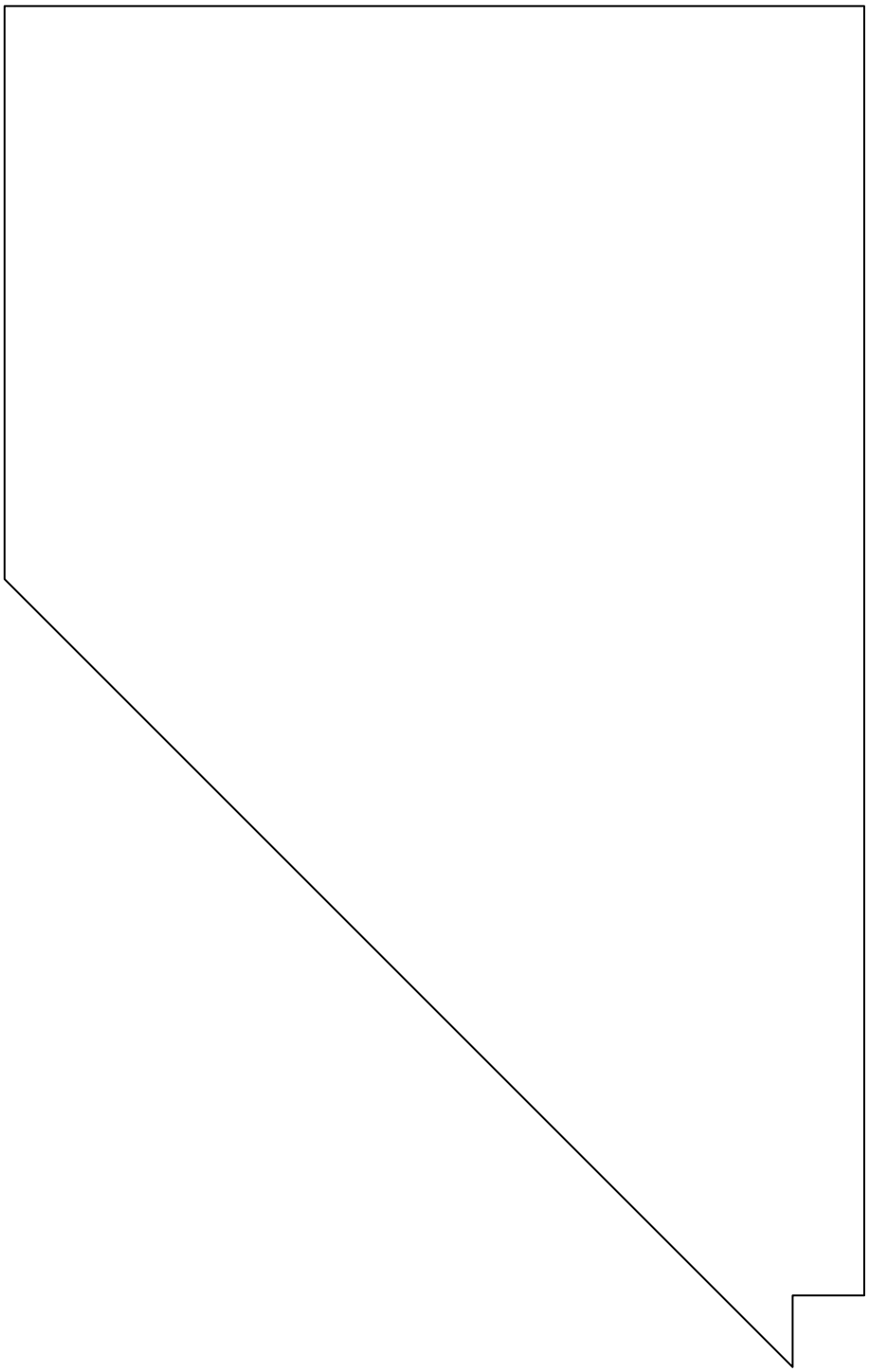
Geoboard paper



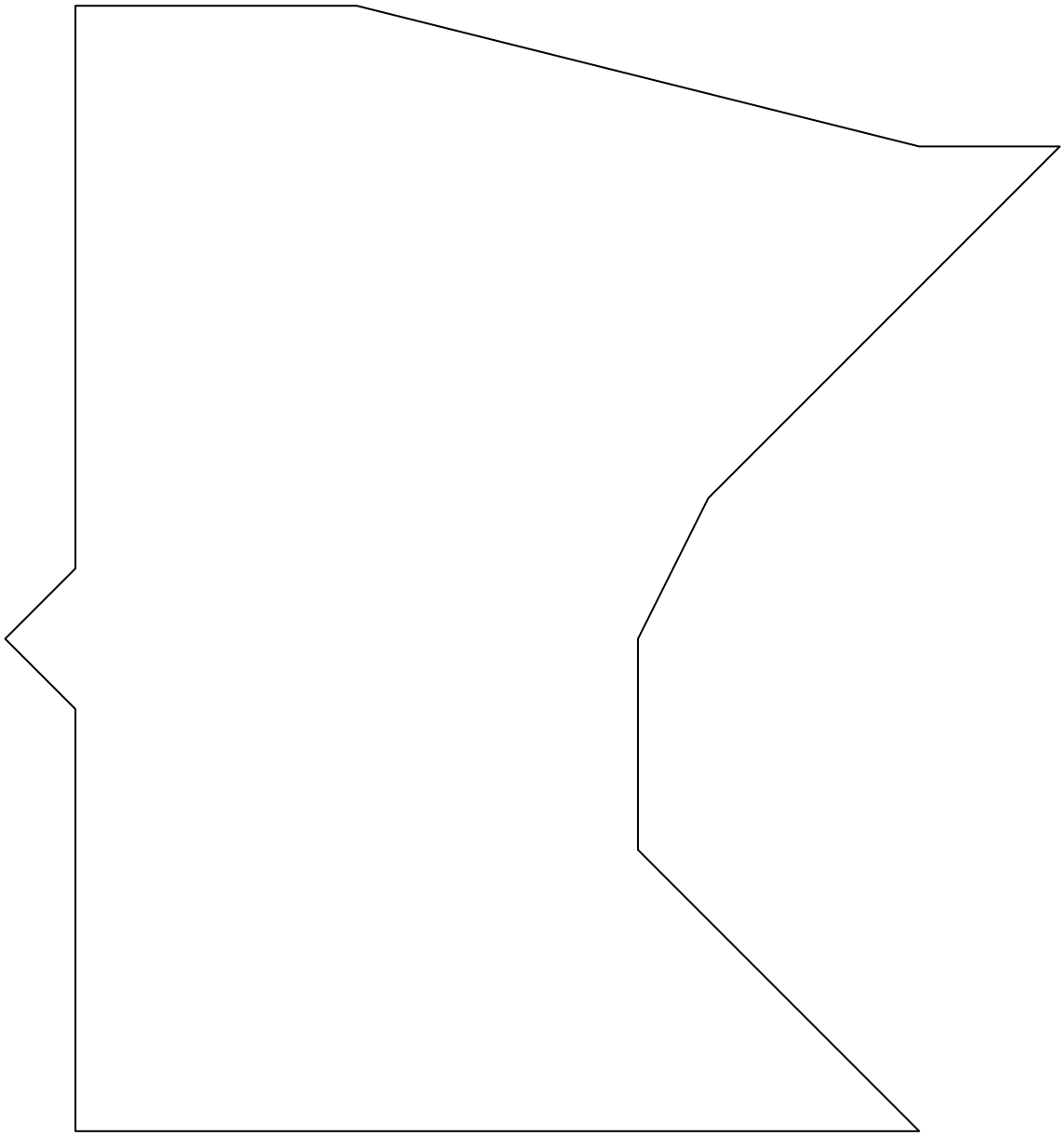
Geoboard display master



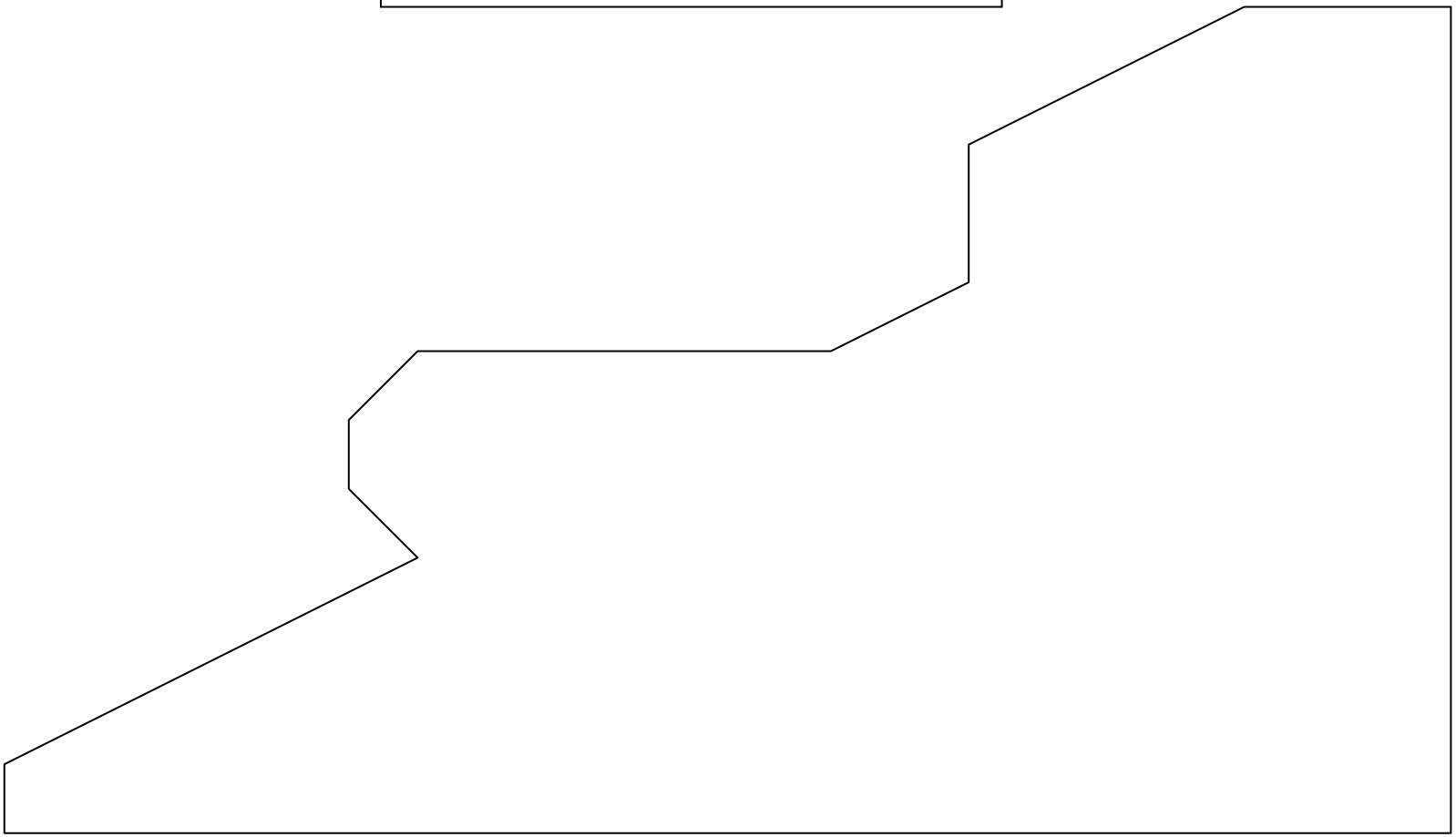
Nevada
1 square cm = 660.03 square mi.



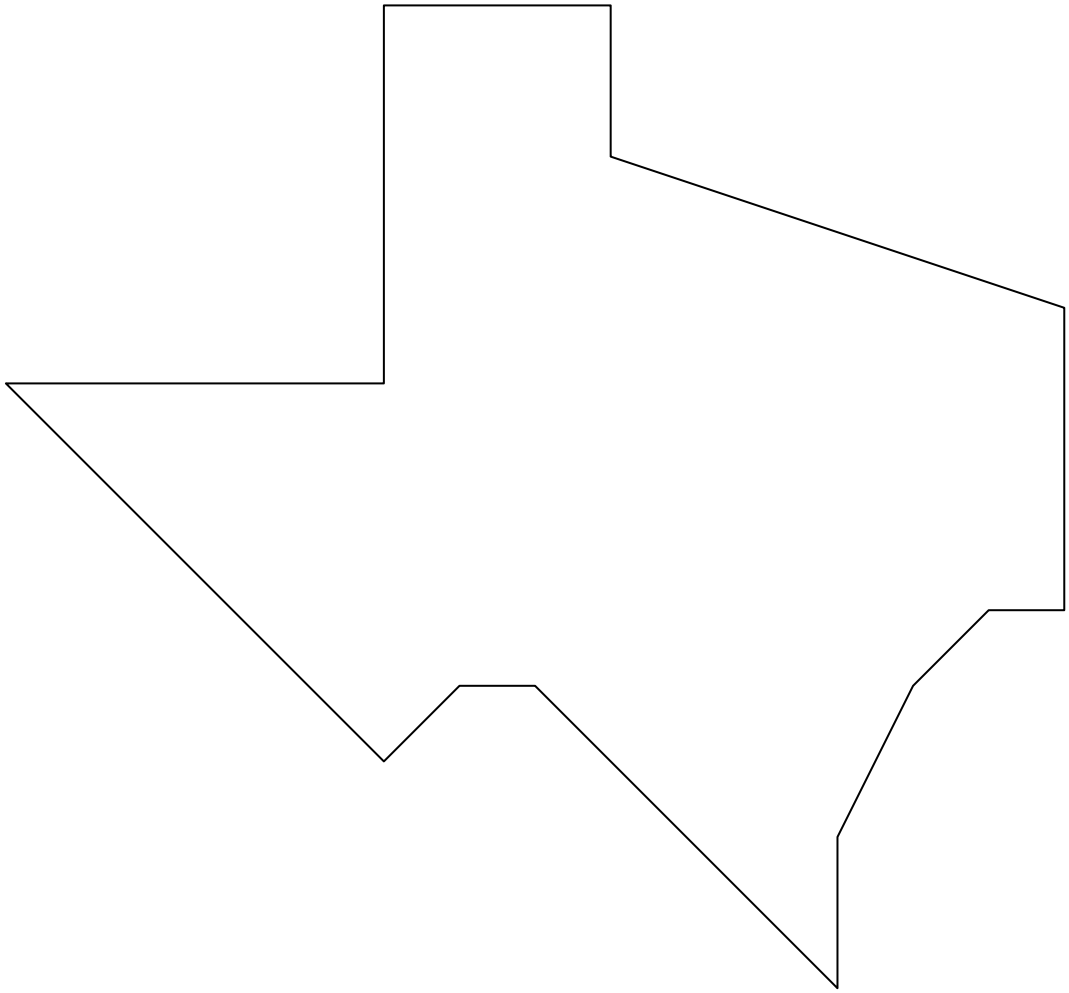
Minnesota
1 square cm = 515.32 square mi.



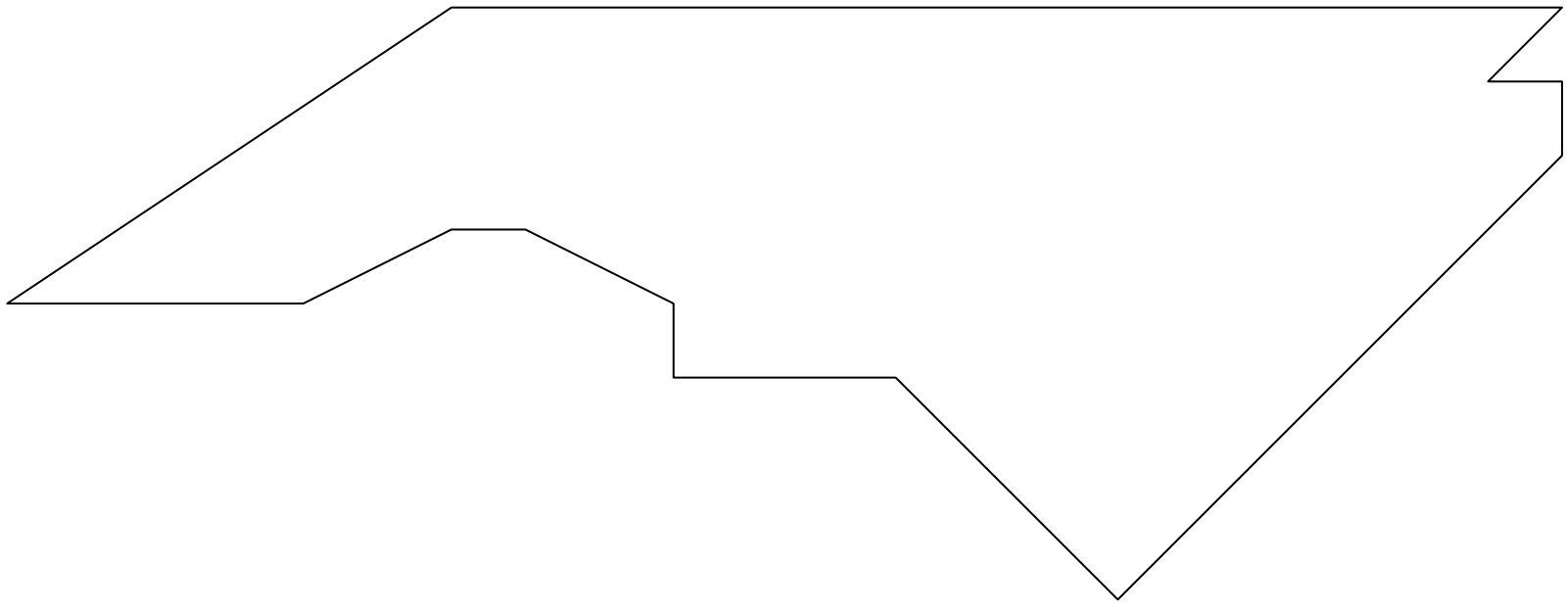
New York
1 square cm = 306.65 square mi.



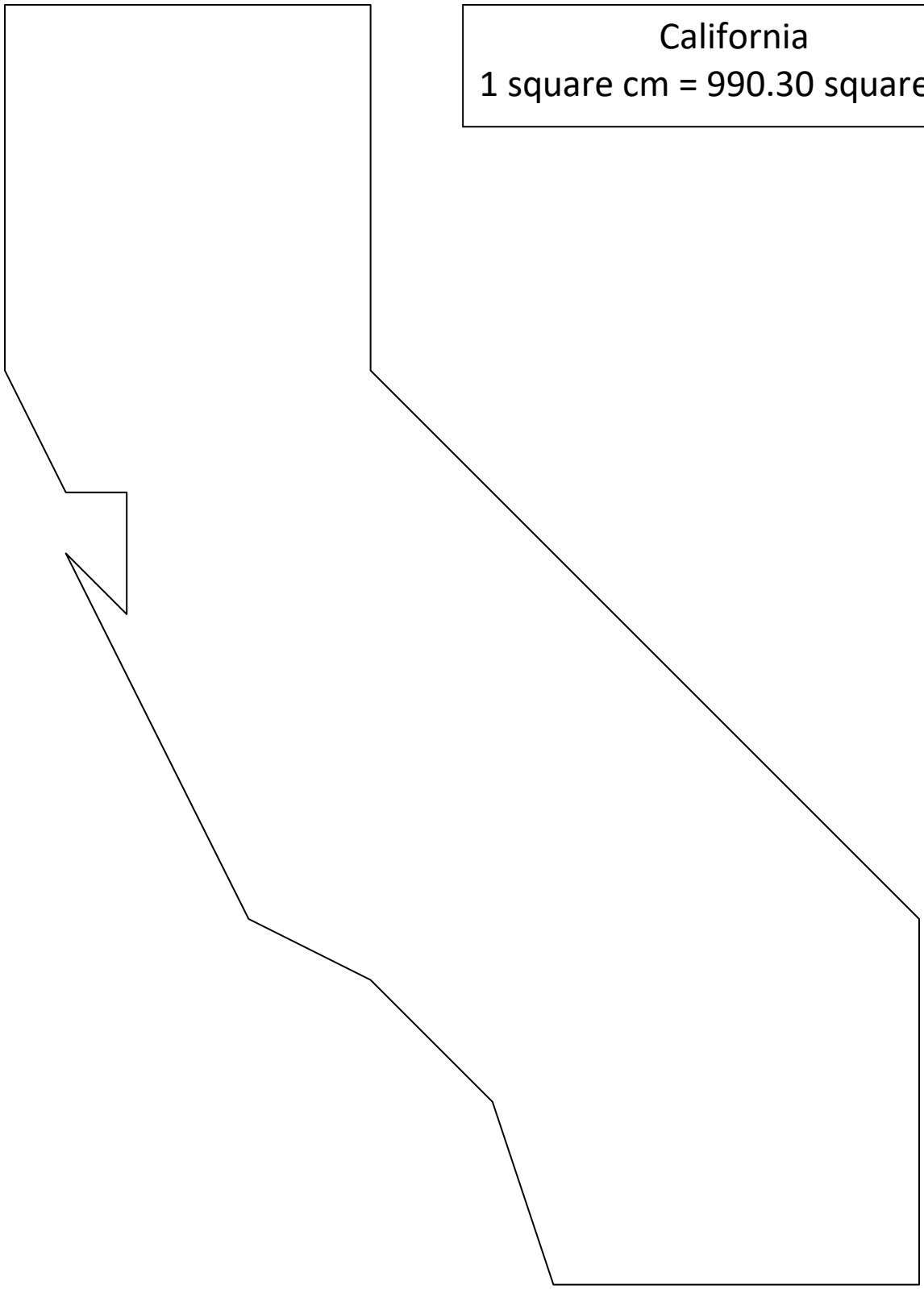
Texas
1 square cm = 3273.925 square mi.



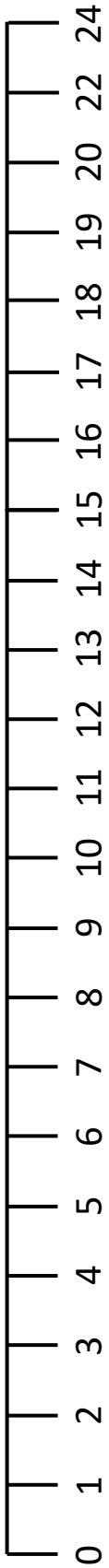
North Carolina
1 square cm = 559.98 square mi.

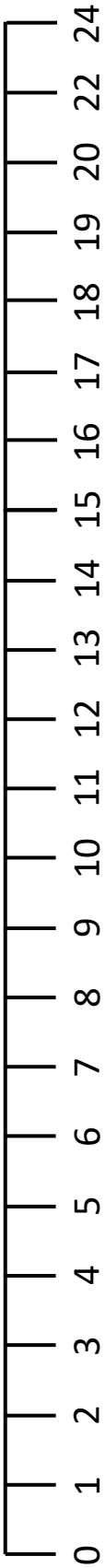


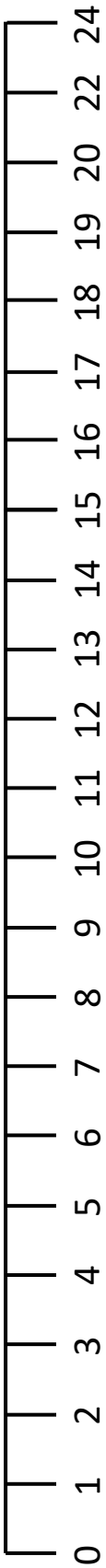
California
1 square cm = 990.30 square mi.

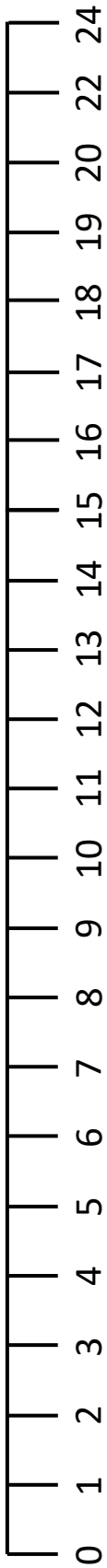


Centimeter rulers









Tangram Math

Integrating Fractions, Decimals, Percent, Geometry, and Algebra

Overview:

In this powerfully engaging activity students of all skill levels will study standard and nonstandard tangrams to determine the values of the pieces. Students will compare the pieces to see how they relate to one another as fractions, decimals, percent, and areas. They will also develop geometric vocabulary and form an understanding of congruence and similarity. The best part is that the entire process will lead them seamlessly into algebraic thinking as they navigate among the pieces and their representations. The activity can also be used to help students learn probability, solve proportions and equations and understand algebraic properties!

Required Materials:

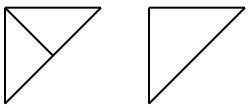
- ☒ Student copies of the tangrams

Optional Materials:

- ☐ Rulers
- ☐ Scissors
- ☐ Calculators
- ☐ Document camera or projection device

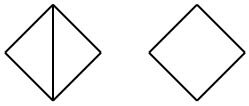
Procedure:

1 This activity works best when students work in groups of two to four. This fosters important dialogue that facilitates understanding. Display a copy of tangram 1. Students should have individual copies. You may wish to distribute scissors for this activity as some students find it helpful to cut the pieces for comparison.



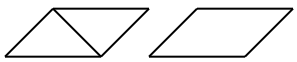
2c = b

2 Ask the students, "If the entire square tile has a value of 1, what is the value of the region *a*?" They can see that it is $^1/_4$ since four of the large triangles can fit in the square.



2c = d

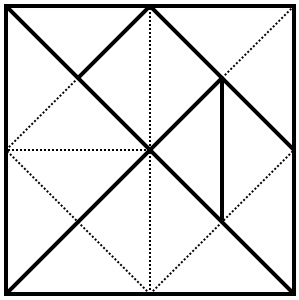
3 Next ask them to evaluate the medium-sized triangle, region b. Since two b's will fit into one a, b is $^1/_2$ of a, or $^1/_8$ of the whole. Another way to see this is by showing that the entire tile can be cut into eight b's.



2c = e

4 Ask the students to find the values of the other regions. When they find answers, ask them to justify them. They may do this verbally, by rearranging pieces on their desks or on your projection device. They will

see that the small triangle c is $^1/_{16}$ of the tile since two of them can fit into b. Since two small triangles also fit into the square, d is equal to b. Two c's also fit into the parallelogram e, so b, d, and e are all $^1/_8$. This is shown in the margin on the previous page.



5 Another way to show the relationships among the polygons is by drawing lines to subdivide the tangram into the smallest unit (in this case, triangle c) as shown here in the margin. It is then easy to see that c is $^1/_{16}$ of the tile. Regions b, d, and e are each $^2/_{16}$ or $^1/_8$ of the tile, and region a is $^4/_{16}$ or $^1/_4$ of the tile. This cut-up method will not work on all of the other tangram patterns however.

6 After the students have found the fractional values of each piece, they can add them together to check their work. Remember that there are two of shape a and two of shape c. Adding all of these pieces gives a total of $^{16}/_{16}$ or 1 (whole tangram).

7 If students are familiar with decimals, you can ask them to find the decimal values of each piece. They will find that a = 0.25. Ask them to explain their reasoning. They may say that in money a quarter is $.25.

8 When they try to find the value of b, students may give different answers. Beginning students fail to see that c, which is $^1/_8$, has a decimal representation of 0.125. They may think that 0.125 is greater than 0.25 since it has more places. I have seen students suggest that b = 0.12½. This may confuse some as it incorporates both decimal and common fractions, but essentially it is correct. We see these sorts of representations on gasoline prices: 2.99^9/_{10}$. By annexing a zero and writing the value of a as 0.250 instead of 0.25, many students are able to halve the 0.250 and get 0.125.

9 I have also seen students suggest that c has a value of 0.625. They annex a zero on b to get 0.1250 and then cut it in half without regard to the place value. These discrepancies will be discovered when students check their answers by adding all the decimals (keeping in mind that there are two a's and two c's). They should get a total value of 1.0000.

10 Next you can ask students to write these as percent representations. This should be much easier than it typically is since students are beginning to see the connections among the shapes and their other representations. In fact, they will see that half of 25% (a) is 12½% (b). Similarly, if they had written that c had a value of 0.625, they will think it has a percent value of 62.5%.

This is a contradiction since they can see that it is not over half the total shape. However, by writing the correct answer of 0.0625, they can then convert that to 6¼%. Adding the percent values of the pieces yields 100%.

11 Students can also calculate the areas of the pieces. Have the students measure the base and height of large triangle a. They will see that the base is four inches long and the height is equal to two inches. Using the area formula for a triangle leads to:

$$A = \frac{4 \bullet 2}{2} = 4\ sq\ in.$$

The students may not need to use these formulas if they make connections with the values of the pieces. For example, since a has an area of 4 in^2, and b = ½ of a, it must have an area of 2 in^2.

12 Ask them to find the areas of the other triangular regions using the same formula. This will show that medium-sized triangle b has a base and height of two inches and an area of two square inches.

$$A = \frac{2 \bullet 2}{2} = 2\ sq.\ in.$$

The small triangle has a base of two inches, a height of one inch, and an area of one square inch.

$$A = \frac{2 \bullet 1}{2} = 1\ sq.\ in.$$

13 The students should now find the area of parallelogram e. They can see that it is composed of two small triangles (c), so its area must be two square inches. Measuring its base and height shows they are equal to two inches and one inch respectively. Multiplying these gives us the area:

$$A = 2 \bullet 1 = 2\ sq.\ in.$$

14 The square will be more difficult to solve. Measuring the side shows it to be approximately $1^3/_8$ inches. Squaring this shows that the area is $1^{57}/_{64}$ square inches. This is almost two square inches. The Pythagorean theorem is more accurate in this case. The sides of the square will represent legs a and b of a right triangle. Since the sides are equal, we will use only the letter a. The diagonal of the square is the hypotenuse, c. Since measuring the hypotenuse shows that it is two inches long, the Pythagorean Theorem is written:

$$a^2 + b^2 = c^2$$
$$a^2 + a^2 = 2^2$$
$$2a^2 = 2^2$$
$$2a^2 = 4$$
$$a^2 = 2 \ (\textit{the area of the square})$$
$$a = \sqrt{2} \approx 1.414 \ in.$$

15 Students should also write the names of each shape. Younger students may want to label shape a with the name triangle. I have my middle school students use the full name: isosceles right triangle. Similarly, some students will say that shape d is a diamond. Others may opt for the more formal term, rhombus. However, the most specific name is square. The fact that it has been rotated from a familiar orientation causes many students to assume it is no longer a square. This is because most squares they have seen have a base parallel to the edges of their paper. When students change the name of a shape when it is rotated, it shows that they are not functioning at a high level of geometric thinking. They assume that shapes are defined by orientation instead of by their properties. Since the formal definition of a square is a quadrilateral with four congruent sides and four congruent angles, d is a square.

> Good Tip
> This activity even can be used as a formative assessment tool! When students talk about their thinking, their vocabulary will show the level of sophistication in their thinking as explained here.

Remind them that triangles are always described by their angles and their sides. There are three classifications of each:

<u>angles</u>
acute – all three angles are less than 90º
right – one angle is exactly 90º
obtuse – one angle is more than 90º
<u>sides</u>
equilateral – all three sides are equal
isosceles – two sides are equal
scalene – no sides are equal

16 Notice that as students discuss their thinking in their group, they will be talking algebraically! You will hear statements such as, “Two c’s is a d,” and, “D is equal to b.” These can then be written as algebraic equations: 2c = d and d = b. I require my students to write an equation for

each shape. The equation must be correct and must contain the letter for that region. For example, these five equations could be used for Tangram 1:

Region	Equation
a	$4c = \mathbf{a}$
b	$\mathbf{b} = 2d$
c	$a/4 = \mathbf{c}$
d	$\mathbf{d} = e$
e	$2a = b + 2c + d + \mathbf{e}$

17 If you want to explore proportions, assign a new value to the tangram. For example, if the entire tangram has a value of \$3.00, what is the value of a? Since a = ¼ of the total, it has a value of \$.75.

If b has a value of ½, what is the value of the entire tangram? Since 8 b's can fit in the tangram, the total value of the tangram is 8 • ½ = 4.

18 Advanced students can calculate the perimeters of the pieces. However, this will require the use of the Pythagorean theorem or very accurate measurement. Each tangram has a side length of 4 inches. Therefore the legs of triangle b are each 2" long. The Pythagorean theorem gives the length of the hypotenuse as:

$$a^2 + b^2 = c^2$$
$$2^2 + 2^2 = c^2$$
$$4 + 4 = c^2$$
$$8 = c^2$$
$$2.83 \approx c$$
$$\text{Perimeter} = a + b + c \approx 2 + 2 + 2.83 = 6.83$$

19 Have students explore the other tangram patterns. The first six involve fractions that are based on halves such as $1/4$, $1/8$, and $1/16$. Halves are easier to work with conceptually than thirds or fifths. For this reason, only the last two tangrams involve thirds or fifths. Also, in the first three tangrams, all areas use unit fractions; that is, their numerators are one. This changes in tangrams four, five, and six where we start to see fractions such as $3/8$. Again this represents another conceptual step in part/whole thinking.

Journal Prompts:

Explain how polygons b, d, and e are alike and how they are different.

Why does the cut-up method only work on some tiles but not on others?

Homework:

You may wish to assign an unfinished tangram as homework.

Another option is to have students make a tile of their own and write fractions for each region. It should be made of at least seven sections using at least five different shapes. This should be drawn on a 4” or a 6" square. You may wish to pass out the included grid paper for this task.

Taking a Closer Look:

Assign a cost or value to various regions. If the medium-sized triangle of tangram one sells for $1.37, what is the cost of each region and the whole tile? If tangram two costs $5.64, what would each piece cost?

Explore probability. What is the probability of a dart randomly landing on c?

P(c) = $^1/_{16}$

What is the probability of a dart randomly hitting a quadrilateral?

P(quadrilateral) = d + e = $^1/_8 + ^1/_8 = ¼$

Assessment:

By allowing students to work in groups and by asking them to rationalize their answers, you will be able to assess their levels of understanding. Listen to their discussions as mentioned previously. Are they using sophisticated reasoning and accurate vocabulary?

Homework can be assessed after collecting it using the enclosed answer keys, or you may wish to have students trade papers and solve each other’s puzzles.

Answer Key

Tangram 1

Region	Name	Frac.	Dec.	%	Area
a	Isosceles right triangle	1/4	0.25	25	4 in^2
b	Isosceles right triangle	1/8	0.125	12.5	2 in^2
c	Isosceles right triangle	1/16	0.0625	6.25	1 in^2
d	Square	1/8	0.125	12.5	2 in^2
e	Parallelogram	1/8	0.125	12.5	2 in^2

Tangram 2

Region	Name	Frac.	Dec.	%	Area
a	Isosceles right triangle	1/4	0.25	25	4 in^2
b	Isosceles right triangle	1/16	0.0625	6.25	1 in^2
c	Isosceles right triangle	1/8	0.125	12.5	2 in^2
d	Rectangle	1/4	0.25	25	4 in^2
e	Scalene right triangle	1/16	0.0625	6.25	1 in^2
f	Obtuse isosceles triangle	1/8	0.125	12.5	2 in^2

Tangram 3

Region	Name	Frac.	Dec.	%	Area
a	Rectangle	1/8	0.125	12.5	2 in^2
b	Scalene right triangle	1/16	0.0625	6.25	1 in^2
c	Square	1/16	0.0625	6.25	1 in^2
d	Rectangle	1/32	0.03125	3.125	.5 in^2
e	Acute isosceles triangle	1/8	0.125	12.5	2 in^2
f	Rhombus	1/4	0.25	25	4 in^2
g	Obtuse isosceles triangle	1/32	0.03125	3.125	.5 in^2
h	Acute isosceles triangle	1/32	0.03125	3.125	.5 in^2

Tangram 4

Region	Name	Frac.	Dec.	%	Area
a	Isosceles right triangle	1/4	0.25	25	4 in^2
b	Isosceles right triangle	1/8	0.125	12.5	2 in^2
c	Right trapezoid	1/8	0.125	12.5	2 in^2
d	Isosceles right triangle	1/16	0.0625	6.25	1 in^2
e	Isosceles trapezoid	3/16	0.1875	18.75	3 in^2

Tangram 5

Region	Name	Frac.	Dec.	%	Area
a	Rectangle	1/8	0.125	12.5	2 in^2
b	Scalene right triangle	1/16	0.0625	6.25	1 in^2
c	Isosceles right triangle	1/16	0.0625	6.25	1 in^2
d	Isosceles right triangle	1/32	0.03125	3.125	.5 in^2
e	Isosceles trapezoid	3/32	0.09375	9.375	1.5 in^2
f	Right trapezoid	3/8	0.375	37.5	6 in^2

Tangram 6

Region	Name	Frac.	Dec.	%	Area
a	Rectangle	1/8	0.125	12.5	2 in^2
b	Acute isosceles triangle	1/16	0.0625	6.25	1 in^2
c	Acute right triangle	1/32	0.03125	3.125	.5 in^2
d	Rectangle	1/8	0.125	12.5	2 in^2
e	Isosceles trapezoid	3/16	0.1875	18.75	3 in^2
f	Scalene right triangle	1/16	0.0625	6.25	1 in^2
g	Acute isosceles triangle	1/8	0.125	12.5	2 in^2
h	Parallelogram	3/16	0.1875	18.75	3 in^2

Tangram 7

Region	Name	Frac.	Dec.	%	Area
a	Rectangle	1/3	$0.\overline{3}$	$33^1/_3$	$5^1/_3$ in^2
b	Obtuse isosceles triangle	1/6	$0.1\overline{6}$	$16^2/_3$	$2^2/_3$ in^2
c	Scalene right triangle	1/12	$0.08\overline{3}$	$8^1/_3$	$1^1/_3$ in^2
d	Rectangle	1/6	$0.1\overline{6}$	$16^2/_3$	$2^2/_3$ in^2
e	Rectangle	1/12	$0.08\overline{3}$	$8^1/_3$	$1^1/_3$ in^2
f	Scalene right triangle	1/24	$0.041\overline{6}$	$4^1/_6$	$^2/_3$ in^2

Tangram 8

Region	Name	Frac.	Dec.	%	Area
a	Rectangle	1/5	0.2	20	3.2 in^2
b	Scalene right triangle	1/10	0.1	10	1.6 in^2
c	Obtuse isosceles triangle	1/20	0.05	5	0.8 in^2
d	Acute isosceles triangle	1/20	0.05	5	0.8 in^2
e	Scalene right triangle	1/20	0.05	5	0.8 in^2
f	Parallelogram	1/10	0.1	10	1.6 in^2
g	Rectangle	1/20	0.05	5	0.8 in^2
h	Acute isosceles triangle	1/10	0.1	10	1.6 in^2
i	Concave pentagon	1/5	0.2	20	3.2 in^2

Look at all the Common Core Standards you can teach with tangrams!

4th grade:

CCSS.MATH.CONTENT.4.NF.A.1
Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

CCSS.MATH.CONTENT.4.NF.A.2
Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

CCSS.MATH.CONTENT.4.NF.B.3
Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

CCSS.MATH.CONTENT.4.NF.B.3.A
Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

CCSS.MATH.CONTENT.4.NF.B.3.B
Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. *Examples: 3/8 = 1/8 + 1/8 + 1/8 ; 3/8 = 1/8 + 2/8 ; 2 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8*.

CCSS.MATH.CONTENT.4.G.A.2
Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

5th grade:

CCSS.MATH.CONTENT.5.NBT.A.1
Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.

CCSS.MATH.CONTENT.5.NBT.A.3
Read, write, and compare decimals to thousandths.

CCSS.MATH.CONTENT.5.NBT.A.3.A
Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., $347.392 = 3 \times 100 + 4 \times 10 + 7 \times 1 + 3 \times (1/10) + 9 \times (1/100) + 2 \times (1/1000)$.

CCSS.MATH.CONTENT.5.NBT.A.3.B
Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.

CCSS.MATH.CONTENT.5.NF.A.1
Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)*

CCSS.MATH.CONTENT.5.NF.A.2
Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. *For example, recognize an incorrect result 2/5 + 1/2 = 3/7, by observing that 3/7 < 1/2.*

CCSS.MATH.CONTENT.5.G.B.3
Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.

6th grade:

CCSS.MATH.CONTENT.6.RP.A.1
Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*
CCSS.MATH.CONTENT.6.RP.A.3
Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

CCSS.MATH.CONTENT.6.RP.A.3.C
Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

CCSS.MATH.CONTENT.6.RP.A.3.D
Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

CCSS.MATH.CONTENT.6.NS.C.6
Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

CCSS.MATH.CONTENT.6.EE.A.3
Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3 (2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6 (4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

CCSS.MATH.CONTENT.6.EE.A.4
Identify when two expressions are equivalent (i.e., when the two expressions name the same

number regardless of which value is substituted into them). *For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.*

CCSS.MATH.CONTENT.6.EE.B.6
Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

CCSS.MATH.CONTENT.6.G.A.1
Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

7th grade:

CCSS.MATH.CONTENT.7.RP.A.2
Recognize and represent proportional relationships between quantities.

CCSS.MATH.CONTENT.7.RP.A.2.C
Represent proportional relationships by equations. *For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as t = pn.*

CCSS.MATH.CONTENT.7.NS.A.2

Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

CCSS.MATH.CONTENT.7.NS.A.2.D
Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

CCSS.MATH.CONTENT.7.EE.B.4
Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

CCSS.MATH.CONTENT.7.G.B.6
Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

8th grade:

CCSS.MATH.CONTENT.8.NS.A.2
Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g.,

π^2). *For example, by truncating the decimal expansion of √2, show that √2 is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations*.

CCSS.MATH.CONTENT.8.NS.A.1
Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually

CCSS.MATH.CONTENT.8.EE.A.2
Use square root and cube root symbols to represent solutions to equations of the form$x^2 = p$ and x^3 = p, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that √2 is irrational.

CCSS.MATH.CONTENT.8.G.A.2
Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

CCSS.MATH.CONTENT.8.G.A.4
Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.

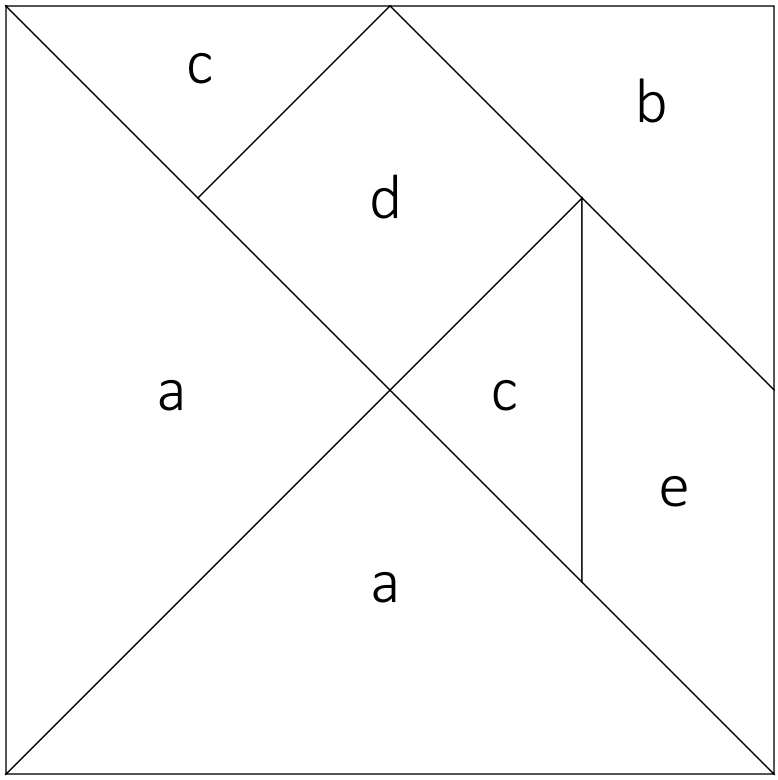
CCSS.MATH.CONTENT.8.G.A.5
Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so*.

CCSS.MATH.CONTENT.8.G.B.7
Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

Tangram 1

Name_____

Date_____ Class_____

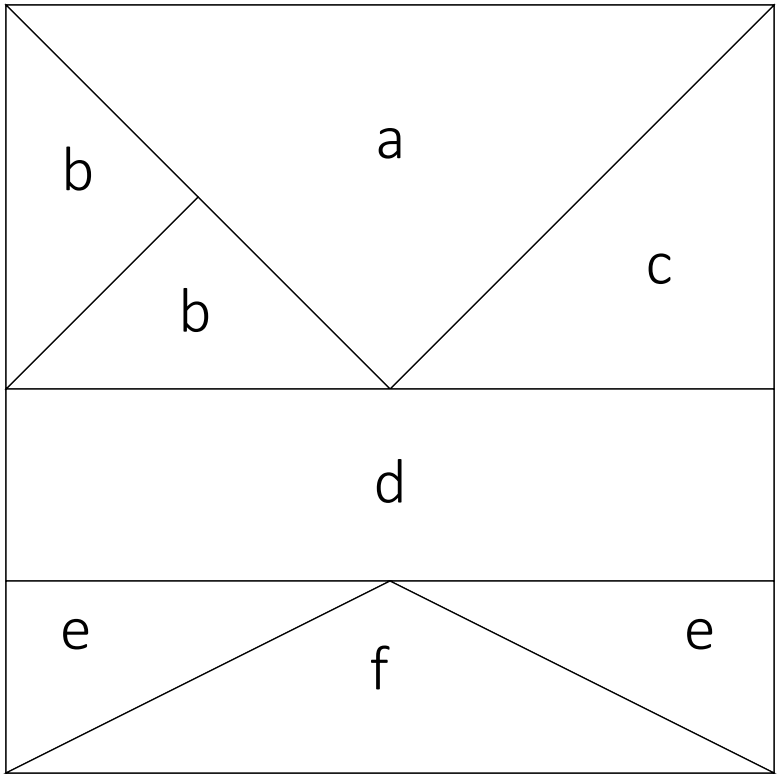


Region	Name	Fraction	Decimal	Percent	Area	Formula
a	_____	_____	_____	_____	_____	_____
b	_____	_____	_____	_____	_____	_____
c	_____	_____	_____	_____	_____	_____
d	_____	_____	_____	_____	_____	_____
e	_____	_____	_____	_____	_____	_____

Tangram 2

Name_____

Date_____ Class_____

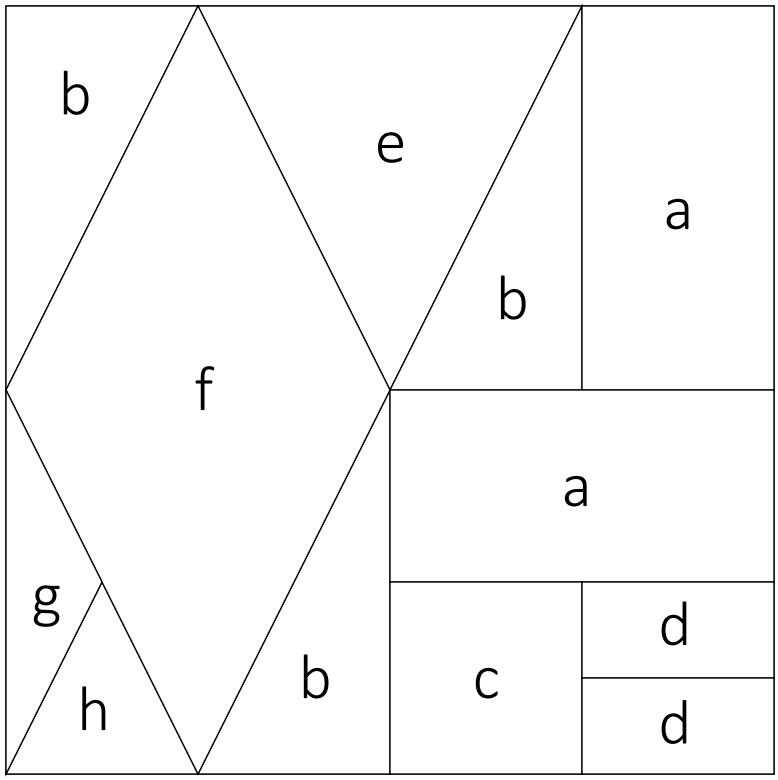


Region	Name	Fraction	Decimal	Percent	Area	Formula
a	_____	_____	_____	_____	_____	_____
b	_____	_____	_____	_____	_____	_____
c	_____	_____	_____	_____	_____	_____
d	_____	_____	_____	_____	_____	_____
e	_____	_____	_____	_____	_____	_____
f	_____	_____	_____	_____	_____	_____

Tangram 3

Name_____

Date_____ Class_____

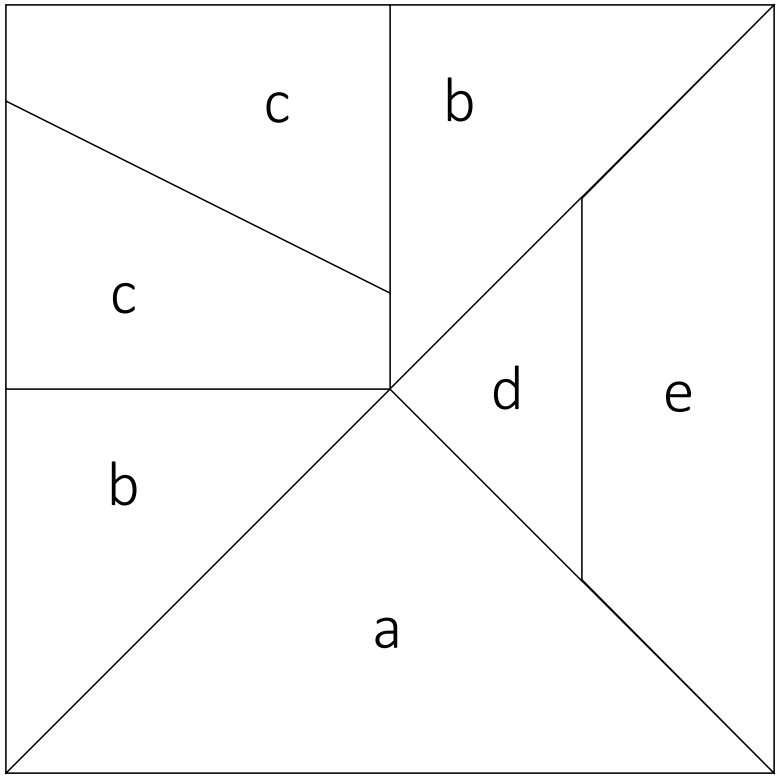


Region	Name	Fraction	Decimal	Percent	Area	Formula
a	_____	_____	_____	_____	_____	_____
b	_____	_____	_____	_____	_____	_____
c	_____	_____	_____	_____	_____	_____
d	_____	_____	_____	_____	_____	_____
e	_____	_____	_____	_____	_____	_____
f	_____	_____	_____	_____	_____	_____
g	_____	_____	_____	_____	_____	_____
h	_____	_____	_____	_____	_____	_____

Tangram 4

Name_____

Date_____ Class_____

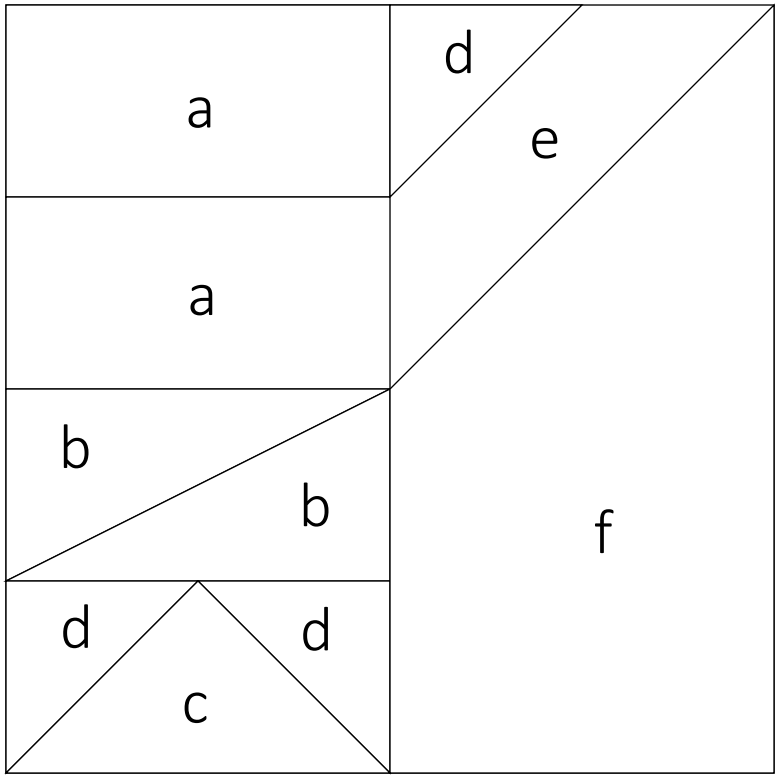


Region	Name	Fraction	Decimal	Percent	Area	Formula
a	_____	_____	_____	_____	_____	_____
b	_____	_____	_____	_____	_____	_____
c	_____	_____	_____	_____	_____	_____
d	_____	_____	_____	_____	_____	_____
e	_____	_____	_____	_____	_____	_____

Tangram 5

Name_____

Date_____ Class_____

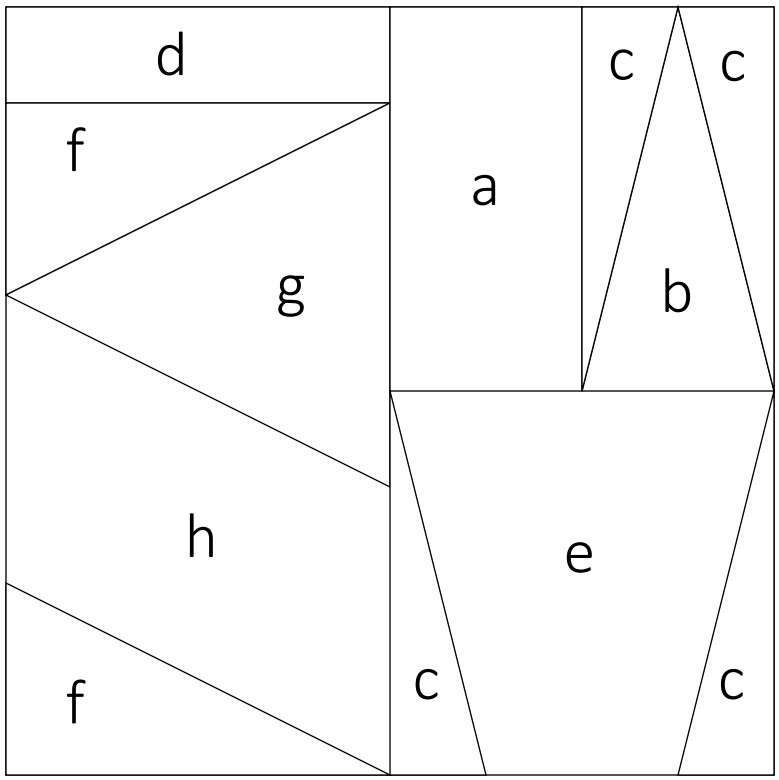


Region	Name	Fraction	Decimal	Percent	Area	Formula
a	_____	_____	_____	_____	_____	_____
b	_____	_____	_____	_____	_____	_____
c	_____	_____	_____	_____	_____	_____
d	_____	_____	_____	_____	_____	_____
e	_____	_____	_____	_____	_____	_____
f	_____	_____	_____	_____	_____	_____

Tangram 6

Name_____

Date_____ Class_____

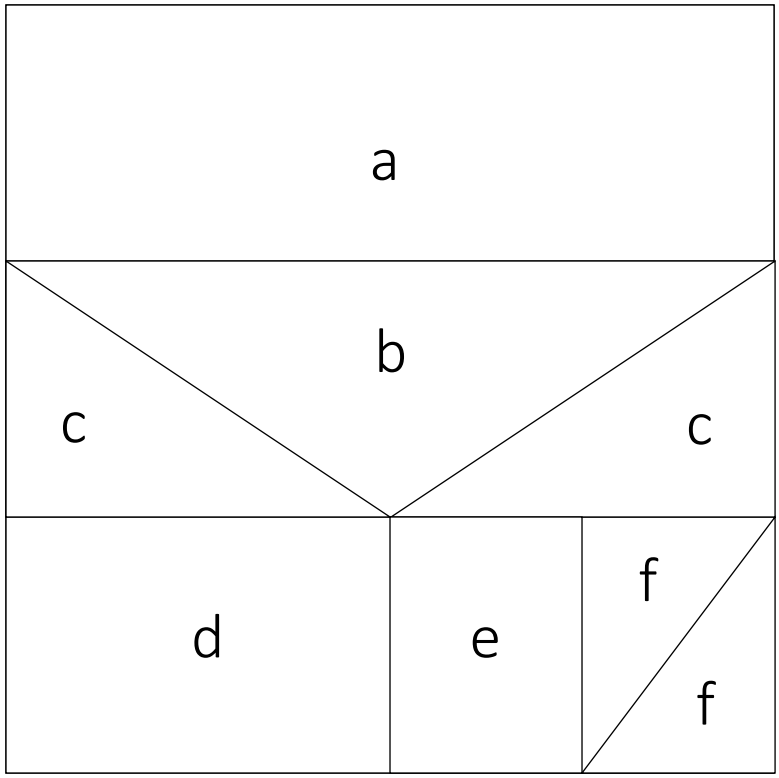


Region	Name	Fraction	Decimal	Percent	Area	Formula
a	_____	_____	_____	_____	_____	_____
b	_____	_____	_____	_____	_____	_____
c	_____	_____	_____	_____	_____	_____
d	_____	_____	_____	_____	_____	_____
e	_____	_____	_____	_____	_____	_____
f	_____	_____	_____	_____	_____	_____
g	_____	_____	_____	_____	_____	_____
h	_____	_____	_____	_____	_____	_____

Tangram 7

Name_____

Date_____ Class_____

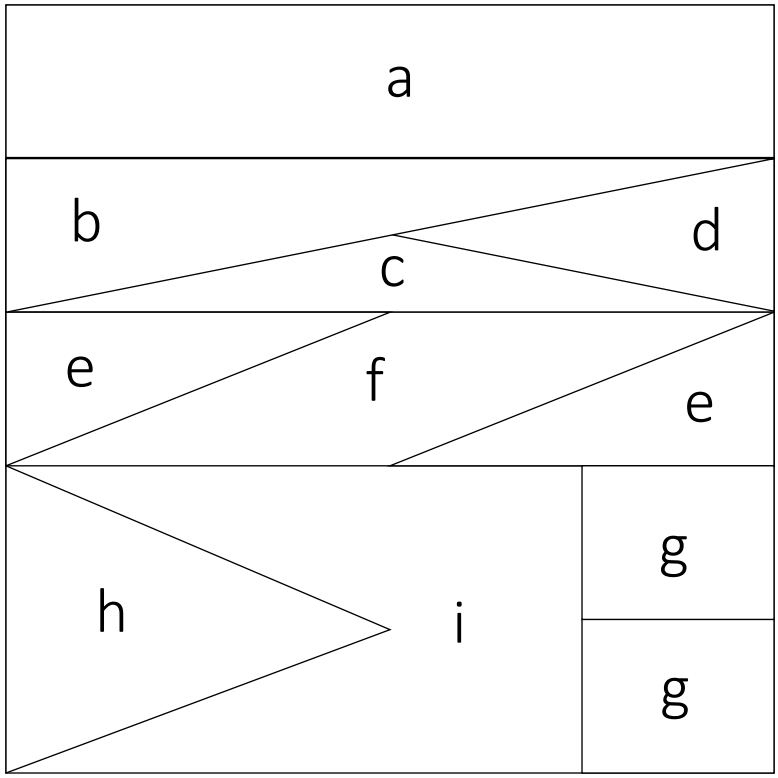


Region	Name	Fraction	Decimal	Percent	Area	Formula
a	_____	_____	_____	_____	_____	_____
b	_____	_____	_____	_____	_____	_____
c	_____	_____	_____	_____	_____	_____
d	_____	_____	_____	_____	_____	_____
e	_____	_____	_____	_____	_____	_____
f	_____	_____	_____	_____	_____	_____

Tangram 8

Name_____

Date_____ Class_____



Region	Name	Fraction	Decimal	Percent	Area	Formula
a						
b						
c						
d						
e						
f						
g						
h						
i						

My Personal Tangram

Name_____

Date_____ Class_____

Create a tangram of your own design using at least five different shapes and seven different pieces as in Tangram 1. Use your creativity. Assign a value of 1 to your tangram, and then find the values of each piece.

Region	Name	Fraction	Decimal	Percent	Area	Formula
____	_____	_____	_____	_____	_____	_____
____	_____	_____	_____	_____	_____	_____
____	_____	_____	_____	_____	_____	_____
____	_____	_____	_____	_____	_____	_____
____	_____	_____	_____	_____	_____	_____
____	_____	_____	_____	_____	_____	_____
____	_____	_____	_____	_____	_____	_____
____	_____	_____	_____	_____	_____	_____